The Power of Tests for Pareto Efficiency Within the Family

Jesse Naidoo*
University of Chicago

JOB MARKET PAPER

This Version: December 10, 2014

Please click HERE for the latest version of this paper.

JEL codes: D13, D7.

Abstract

Empirical studies of the allocation of consumption within the family have often failed to reject the null hypothesis of Pareto efficiency. To study the power of these tests, I construct a class of noncooperative models with inefficient equilibria. In these models, variations in the economic or social circumstances outside of the household ("distribution factors") cause variations in the distribution of wealth within the household. If these implicit transfers go unobserved by the econometrician, noncooperative models imply the same restrictions on household demand as do efficient models. Therefore, many tests in the literature cannot identify inefficiency when it is present. I then show that without restrictions on preferences or data on individual consumption, the hypothesis of Pareto efficiency is not well-separated from the noncooperative alternative. So in a nonparametric setting, there can be no tests that both reject all inefficient models and do *not* reject any efficient models. However, if the relative wealth of each partner is observed without error and satisfies a large support condition, and if the consumption of a public good and at least one assignable private good can be observed, then it is possible to reject efficiency.

* *email:* **jnaidoo@uchicago.edu**. I thank Azeem Shaikh, Alessandra Voena, Robert Garlick, Chen Yeh, Elisa Olivieri, Alex Frankel, Stephane Bonhomme, Magne Mogstad, and especially Derek Neal for their comments and for their support. All errors are my own.

1 Introduction

A glance at Table 1 creates the impression that, although the literature is short of a consensus, Pareto efficiency characterizes family life well. That is, many studies have failed to reject the null hypothesis of efficiency. What remains unexamined, though, is whether those tests have power against inefficient alternatives - or, to put the question more sharply, whether those tests have the ability to identify *in*efficiency.

Table 1: Selected empirical studies of allocation within the family. A "unitary" model of the family is one in which there exists a positive representative consumer.

Paper	**Country**	**Subject**	**Conclusions**
[Attanasio & Lechene, 2014]	Mexico	consumer goods	favors efficiency
[Bayudan, 2006]	Philippines	time use	favors efficiency
[Bobonis, 2009]	Mexico	consumer goods	favors efficiency
[Bourguignon *et al.* , 1993]	France	consumer goods	favors efficiency
[Browning *et al.* , 1994]	Canada	consumer goods	favors efficiency
[Browning & Chiappori, 1998]	Canada	consumer goods	favors efficiency
[Chiappori *et al.* , 2002]	US	labor supply	favors efficiency
[Del Boca & Flinn, 2014]	US	time use	favors efficiency
[Del Boca & Flinn, 2012]	US	time use	favors efficiency
[Donni, 2007]	France	consumer goods	favors efficiency
[Donni & Moreau, 2007]	France	labor supply	favors efficiency
[Duflo, 2003]	South Africa	child health	rejects unitary model
[Fortin & Lacroix, 1997]	Canada	labor supply	favors efficiency
[Goldstein & Udry, 2008]	Ghana	land use	rejects efficiency
[Lise & Yamada, 2014]	Japan	risk-sharing, credit	rejects ex-ante efficiency
[Lundberg *et al.* , 1997]	UK	consumer goods	rejects unitary model
[Mazzocco, 2007]	US	risk-sharing, credit	rejects ex-ante efficiency
[Vermeulen, 2005]	Netherlands	labor supply	favors efficiency
[Voena, 2010]	US	risk-sharing, credit	rejects ex-ante efficiency
[Udry, 1996]	Burkina Faso	land use	rejects efficiency

To explore the possibility of a false negative, I construct two models of the allocation of goods within a two-person family. The first of these models leads to efficient outcomes, while the second leads to inefficient ones. But, in Section 4, I show that both classes of models lead to a particular proportionality condition. Since the failure to reject this condition has been widely interpreted as evidence for efficiency, this calculation shows that a popular style of test for efficiency cannot detect inefficiency when it is present.

Next, in Section 5, I turn to the question of whether powerful tests can be constructed. My answer is negative: in Proposition 5, I show that in data with no price variation, there are always household demand systems that are consistent with both efficient models and inefficient models. Within this class, there are *no* properties of household demand that are unique to efficient models, and therefore there can be no tests that simultaneously do not reject any efficient models and *do* reject all inefficient models.

Finally, I use my negative results to reassess the evidence on within-family efficiency and suggest ways to break the identification problem. The suggestions I will be able to make involve either obtaining richer data on income and consumption at the individual, rather than the household level, or using data with variation in relative prices. Alternatively, studying household production, not consumption, may lead to more powerful tests of efficiency.

I am not claiming that families are always and everywhere inefficient. One can offer other arguments for efficiency, such as those of [Becker, 1991], based on the existence of specialization or assortative matching. I am claiming that an analysis of cross-sectional expenditure patterns cannot be informative.

My results do not rely on the existence of unobserved heterogeneity in preferences. Instead, my argument is that models of efficient allocation have a natural alternative - namely, that each family member makes voluntary contributions to a public good.[1] But absent price variation, the data can only tell us about income effects, not substitution effects, and it is the latter which are needed to assess whether inefficiency is present - at least at the margin.

There are at least two reasons to examine the efficiency of the family. First, there is much evidence against the "unitary" model of the family, in which all members agree on how to use the household's resources. That evidence comes from many different contexts, but generally consists of the observation that merely redistributing resources from one family member to another affects how those resources are used. If all members had the same preferences, intrafamily redistributions would have no such effects.[2] Nevertheless, it is possible that efficiency prevails, but this cannot be taken for granted.

Second, any discussion of the policies of a welfare state must be informed by a view, perhaps implicit, of how of families use and distribute their resources. By definition, welfare states provide transfers to their citizens, and many countries do so both in cash and in kind; public education and healthcare are only the most expensive examples of in-kind transfers. As [Becker & Murphy, 1988] point out, both sorts of transfers can be motivated by a concern for the well-being of the next generation. But for a policymaker with those concerns, the two tools may be substitutes. It can only make sense to examine their usefulness together, and doing so entails taking a stand on intrafamily efficiency.

Other economists before me have raised questions about the strength of the evidence for "collective" models of the family. The closest relatives of my work are [Del Boca & Flinn, 2012] and [Del Boca & Flinn, 2014]. In the first of those two papers, the authors use time-use

[1] This sort of inefficient model is in no way exotic or pathological, nor is it especially new: [Chen & Woolley, 2001], [Lundberg & Pollak, 1993], and even earlier [McElroy & Horney, 1981] and [Manser & Brown, 1980] all proposed noncooperative models of the family.

[2] [Duflo, 2003] and [Lundberg *et al.* , 1997] are two prominent examples of this type of study, and [Alderman *et al.* , 1995] is an early summary of the evidence.

data from the 2005 wave of the Panel Study of Income Dynamics (PSID) to estimate both a cooperative and a non-cooperative model of labor supply and housework. They find that over three-quarters of the couples in their sample manage to achieve efficiency, at least in a static sense.

However, their econometric models differ substantially from mine. Theirs is a mixture model, so they attribute unexplained differences in time allocation to unobserved differences across households in preferences, home production technologies, and participation constraints. Most of the papers in this literature (such as those in Table 1) avoid implying a degenerate distribution of the data by invoking measurement error in consumption. To be consistent with this literature, that is also the approach I take here.

[Del Boca & Flinn, 2014] uses the same economic model as [Del Boca & Flinn, 2012], but adjoins it to a simulation of a Gale-Shapley matching algorithm. They do so in order to incorporate marriage market patterns into their examination of the efficiency of time allocation for a sample of American couples from the 2007 wave of the PSID. They conclude that their sample is best characterized by Pareto efficiency within marriage, but do not formally test that claim. ([Del Boca & Flinn, 2012] also do not define or test Pareto efficiency econometrically.) Instead, they show that the distribution of a likelihood ratio statistic is such that efficiency is more the likely characterization of their sample. In this paper, I take a more formal approach to the identification of inefficiency.

[Cherchye *et al.* , 2007] and [Cherchye *et al.* , 2009] are methodologically more distant from this paper, although again the questions they address are similar to mine. They provide revealed-preference conditions on household consumption which, they argue, are necessary and sufficient for Pareto optimality. Again, unlike much of the literature in Table 1, those conditions treat a household's observed purchases as exact data, free of measurement error.

In Section 2 below I begin my analysis, by stating my assumptions about the information available to an imagined econometrician.

2 Econometric Preliminaries

2.1 Data, Present and Missing

I assume the econometrician has access to a dataset consisting of observations of many households. These households themselves consist of two members each; I will refer to the two members as A and B. Later, it will become important that prices do not vary across households, so it may be easiest to imagine that the data is cross-sectional.

Let P be a vector of prices, of dimension $m+1$, for some integer $m \geq 1$. Let Q be a vector of quantities purchased by a given household, so the dimension of Q is also $m+1$. I think of Q as the vector of a given household's *aggregate* consumption. It is often difficult to attribute the expenditures on consumer goods, such as food, to either A or B, and furthermore, some consumption may genuinely be "public," such as expenditures on children. So for now I will assume that the econometrician cannot disaggregate the vector Q into separate accounts for public consumption and for the private consumption of A and B each.

Together, A and B have individual incomes Y_A and Y_B at their disposal. I presume the econometrician does not observe these income (or, in a static context, wealth) variables, but only their sum $Y = Y_A + Y_B$. I consider this reasonable because although market labor earnings are easily assignable to either A or B, there are many assets such as housing or land that may be jointly owned. However, the decision to attribute the returns, or implied rent, to either spouse must be arbitrary, so even in ideal circumstances the relative income of each spouse may be hard to pin down. Moreover, income from self-employment is very often misreported, both in the developed and the developing world; and there is obviously no reason to assume that the reporting errors will cancel out at the household level. (See [Bound *et al.* , 2001], or more recently [Hurst *et al.* , 2014], for discussions of the difficulties of recording self-employment income in the US; [Banerjee & Duflo, 2007] and [Deaton, 1997] discuss those problems as they relate to developing countries.)

Still, at least in a static context, budget balance implies that the total value of consumption, $P \cdot Q$, must be equal to aggregate wealth, Y. So to avoid making the identification problem completely hopeless, I will maintain the assumption that the econometrician can learn the value of Y for each household.

Finally, let Z be a vector of social or economic variables which I will call "distribution factors". These are variables which may affect the consumption of a household, but not its budget set. For example, some authors have used the local sex ratio - understood as a proxy for conditions on the marriage market - as a distribution factor. It is certainly true that variables can be found that do not directly affect the income of, or prices facing, any given household, but are nonetheless correlated with consumption patterns. However, the economic mechanism by which the variables in Z affect the household's choice of Q is, for now, left unstated. Most of what I will have to say concerns how to interpret the empirical relationship between Z and Q.

2.2 Household Demand Systems

I assume that the econometrician has enough data so that the joint distribution of (P, Q, Y, Z) is known with certainty, and therefore so is the conditional mean in the population

$$g(p, y, z) \quad = \quad E[Q|P = p, Y = y, Z = z]. \tag{1}$$

In making this assumption, I am avoiding the complications of inference in finite samples, and focusing only on the logical possibilities for the identification of inefficiency under various assumptions about the domain of (p, y, z) over which g can be observed.

Let $U \subset \mathbb{R}^{m+1}_{++} \times \mathbb{R}_{++}$ and $V \subset \mathbb{R}^K$ be open. U is the domain for prices and household wealth, (p, y); V is the domain for the distribution factors z.

DEFINITION 1. Let $g : U \times V \longrightarrow \mathbb{R}^{m+1}_{++}$ be continuously differentiable and such that, for all $((p, y), z) \in U \times V$,

$$\sum_{i=0}^{m} p_i g_i(p, y, z) \quad = \quad y. \tag{2}$$

Then g is called an *extended aggregate demand system* on $U \times V$. △

Given a pair of open sets (U, V) of appropriate dimensions, let $\mathcal{G}(U, V)$ denote the set of all extended aggregate demand systems on the domain $U \times V$.

In most cross-sectional data, though, all households face the same prices. So it will be important to allow for the possibility that, for some $\overline{p}$, only the conditional distribution of (Q, Y, Z) given $P = \overline{p}$, can be known by the econometrician.

To that end, let $\overline{p}$ be such that for some $y > 0$, $(\overline{p}, y) \in U$ and define the $\overline{p}$-section of $U \times V$ as

$$(U \times V)|_{\overline{p}} \quad = \quad \{(y, z) : (\overline{p}, y) \in U, z \in V\}. \tag{3}$$

By multiplying each quantity $g_i(p, y, z)$ by its price p_i, one puts the quantities represented by $g(p, y, z)$ into expenditure form:

$$\sum_{i=0}^{m} \overline{p}_i g_i(\overline{p}, y, z) \quad = \quad y. \tag{4}$$

The above domain restriction and change of units leads to the following definition:

DEFINITION 2. Let $\overline{p}$ be such that for some $y > 0$, $(\overline{p}, y) \in U$. A *restricted aggregate demand system* on $(U \times V)|_{\overline{p}}$ is a continuously differentiable function $h : (U \times V)|_{\overline{p}} \longrightarrow \mathbb{R}^{m+1}_{++}$ such that for all (y, z)

$$\sum_{i=0}^{m} h_i(y, z) \quad = \quad y. \tag{5}$$

△

I write $\mathcal{H}(U, V|\overline{p})$ for the set of all such functions. At any suitable $\overline{p}$, each extended aggregate demand system $g \in \mathcal{G}(U, V)$ induces a restricted aggregate demand system $h \in \mathcal{H}(U, V|\overline{p})$, in the way indicated by equation (4).

I also assume that only "nondegenerate" demand systems are of interest, in the following sense:

DEFINITION 3. Let $h \in \mathcal{H}(U, V|\overline{p})$ be a restricted aggregate demand system, and define

$$\overline{h}_i \quad = \quad \sup \left\{ \frac{h_i(y, z)}{y} : (y, z) \in (U \times V)|_{\overline{p}} \right\} \tag{6}$$

$$\underline{h}_i \quad = \quad \inf \left\{ \frac{h_i(y, z)}{y} : (y, z) \in (U \times V)|_{\overline{p}} \right\} \tag{7}$$

If h is such that, for all goods $0 \leq i \leq m$,

$$0 < \underline{h}_i \quad < \quad \overline{h}_i < 1 \tag{8}$$

then say that h is *nondegenerate*. $\triangle$

h is degenerate if the budget share of at least one good either (i) does not vary at all in the data (so $\underline{h}_i = \overline{h}_i$), (ii) has vanishing consumption (so $\underline{h}_i = 0$), or (iii) occupies the households' entire budget ($\overline{h}_i = 1$). Thus, imposing that h be nondegenerate seems like a very weak requirement.

3 Two Models of Allocation Within the Family

Below, I construct two parallel formulations of an intrahousehold allocation problem. One model leads to efficient allocations, which, following [Chiappori, 1992], I call a "collective" model of the household. The second model leads to inefficient allocations.

I call the second family of models I construct "Cournot" models, because of their resemblance to that classical model of imperfect competition. Inefficiency exists in equilibrium because both A and B contribute voluntarily and noncooperatively to a public good. Thus, under a Cournot model, the equilibrium consumption of the public good is too low. I need to assume the existence of a public good, otherwise the first welfare theorem implies that noncooperative decision-making is efficient. (Of course, if all goods are private, there are no gains from marriage either.)

3.1 Cournot and Collective Models: Definition

3.1.1 Collective Models

A model of the efficient allocation of goods in a many-person family has two components: (i) a description of the family members' individual preferences over the goods, including a list of which goods are public and which are private, and (ii) a description of which particular efficient allocation will be chosen, and how that "collective" choice varies with the parameters describing the family's environment. In the context of Section 2 above, the family's environment is characterized by its budget set and the distribution factors, i.e. the tuple (p, y, z). Formalizing this, we have the following:

Definition 4. Let $\mathcal{I} \subset \{0, 1, \ldots m\}$ and write $n = |\mathcal{I}|$. By permuting indices, we may assume $\mathcal{I} = \{0, 1, \ldots n-1\}$.

Also let u_A, u_B be weakly increasing and quasiconcave functions on $\mathbb{R}^{m+1}_{++}$, and say $\mu : U \times V \longrightarrow (0, 1)$ is smooth. Given (p, y, z), consider the social planner's problem

$$\begin{aligned} &\max_{(q_{AB}, q_A, q_B)} \mu \cdot u_A(q_{AB}, q_A) + [1 - \mu] \cdot u_B(q_{AB}, q_B) \\ \text{subject to} \quad &\sum_{i=0}^{n-1} p_i q_{AB,i} + \sum_{i=n}^{m} p_i (q_{A,i} + q_{B,i}) \leq y. \end{aligned} \tag{9}$$

If (9) has a unique solution $(q^{**}_{AB}, q^{**}_{A}, q^{**}_{B}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{m+1-n}_{++} \times \mathbb{R}^{m+1-n}_{++}$ for all $(p, y, \mu) \in U \times (0, 1)$, and the function

$$g^{**}(p, y, \mu(p, y, z)) \quad = \quad (q^{**}_{AB}, q^{**}_{A} + q^{**}_{B}) \tag{10}$$

is continuously differentiable on $U \times V$, then the tuple $(\mathcal{I}, u_A, u_B, \mu)$ is called a *collective model with n public goods.* $\triangle$

Given open sets $U \subset \mathbb{R}^{m+1}_{++} \times \mathbb{R}_{++}$ and $V \subset \mathbb{R}^K$, write $\Theta_n(U, V)$ for the set of all collective models with n public goods defined for the domain (U, V), and let

$$\Theta(U, V) \quad = \quad \bigcup_{n=0}^{m+1} \Theta_n(U, V) \tag{11}$$

be the set of all collective models on (U, V). I will use $\theta = (\mathcal{I}, u_A, u_B, \mu)$ to denote a typical collective model in $\Theta(U, V)$.

The function μ is often called the "Pareto weight", and inspection of (9) shows that higher values of μ tilt the household's decisions towards A's preferences. μ is allowed to depend on the economic and social environment of the household, (p, y, z).

EXAMPLE 1. Suppose $m = 2$, so there are three goods in total: a public good q_0 and two private goods q_1, q_2. Assume that goods 1 and 2 are "exclusive", so that A does not consume good 2 and B does not consume good 1. Preferences are

$$u_A(q_0, q_{A1}) \quad = \quad \alpha \log(q_0) + (1 - \alpha) \log(q_{A1}) \tag{12}$$

$$u_B(q_0, q_{B1}) \quad = \quad \beta \log(q_0) + (1 - \beta) \log(q_{B2}) \tag{13}$$

for some $0 < \alpha, \beta < 1$. Normalizing the price of the public good to unity, let $p = (1, p_1, p_2)$ be the price vector, and let household wealth y be given.

For any $\mu \in [0, 1]$, the efficient allocation is

$$q_0 \quad = \quad y \cdot [\mu\alpha + (1 - \mu)\beta] \tag{14}$$

$$q_{A1} \quad = \quad \frac{y}{p_1} \cdot \mu(1 - \alpha) \tag{15}$$

$$q_{A2} \quad = \quad 0 \tag{16}$$

$$q_{B1} \quad = \quad 0 \tag{17}$$

$$q_{B2} \quad = \quad \frac{y}{p_2} \cdot (1 - \mu)(1 - \beta). \tag{18}$$

To complete the description of this collective model, suppose A and B supply labor inelastically, earning wages z_A and z_B, and suppose that the Pareto weight μ is given by

$$\mu(y, z_A, z_B) \quad = \quad \frac{1}{2y}(y + z_A - z_B). \tag{19}$$

A's Pareto weight, μ, depends positively on z_A and negatively on z_B, perhaps because each individual's wages are correlated with their outside option on the marriage market.

The aggregate demands generated by this collective model are

$$\begin{aligned} g_0^{**}(p, y, \mu(y, z_A, z_B)) &= y \cdot [\alpha\mu(y, z_A, z_B) + \beta(1 - \mu(y, z_A, z_B))] \\ &= \alpha \cdot \frac{1}{2}(y + z_A - z_B) + \beta \cdot \frac{1}{2}(y + z_B - z_A) \end{aligned} \tag{20}$$

$$\begin{aligned} g_1^{**}(p, y, \mu(y, z_A, z_B)) &= \frac{y}{p_1} \cdot (1 - \alpha)\mu(y, z_A, z_B) \\ &= \frac{(1-\alpha)}{2p_1} \cdot (y + z_A - z_B) \end{aligned} \tag{21}$$

$$\begin{aligned} g_2^{**}(p, y, \mu(y, z_A, z_B)) &= \frac{y}{p_2} \cdot (1 - \beta)(1 - \mu(y, z_A, z_B)) \\ &= \frac{(1-\beta)}{2p_2} \cdot (y + z_A - z_B) \end{aligned} \tag{22}$$

△

3.1.2 Cournot Models

Under a collective model, variations in the distribution factors z at fixed values of (p, y) cause the household to move along a utility possibility frontier. Those variations along a fixed Pareto frontier cause changes in the aggregate consumption patterns of the household.

However, changes in a family's consumption patterns can be caused by variations in the intrahousehold distribution of resources, too. An empirical relationship between distribution factors z and consumption patterns can also be rationalized by models which allow for distribution factors and the intrahousehold distribution of wealth to be correlated.

EXAMPLE 2. As before, suppose A and B supply labor inelastically, earning wages z_A and z_B. They also have nonlabor wealth z_{AB}, to which they have equal claim. Thus,

$$y = z_A + z_B + z_{AB}. \tag{23}$$

Since each member of the household has equal claim to the "joint" wealth z_{AB}, and full claim to his or her own labor earnings, A's total wealth is

$$\begin{aligned} y_A &= z_A + \frac{1}{2} z_{AB} \\ &= \frac{1}{2}(y + z_A - z_B) \end{aligned} \tag{24}$$

and A's relative wealth is

$$\omega(y, z_A, z_B) = \frac{1}{2y}(y + z_A - z_B). \tag{25}$$

△

A noncooperative model of allocation in a many-person family has similar ingredients to a collective one, namely (i) a description of the preferences of the family members, and (ii) a description of how the distribution of wealth varies with the social and economic environment. I formalize this below.

DEFINITION 5. Let $\mathcal{I} \subset \{0, 1, \dots m\}$ be a singleton. By permuting indices, we may assume $\mathcal{I} = \{0\}$.

Also let u_A, u_B be weakly increasing and quasiconcave functions on $\mathbb{R}^{m+1}_{++}$, and say $\omega : U \times V \longrightarrow (0, 1)$ is smooth. Consider the system

$$\max_{(q_{A0}, q_A)} u_A(q_{A0} + q_{B0}, q_A) \quad \text{s.t.} \quad p_0 q_{A0} + \sum_{i=1}^{m} p_i q_{Ai} \leq y\omega \tag{26}$$

$$\max_{(q_{B0}, q_B)} u_B(q_{A0} + q_{B0}, q_B) \quad \text{s.t} \quad p_0 q_{B0} + \sum_{i=1}^{m} p_i q_{Bi} \leq y(1 - \omega). \tag{27}$$

If the system (26) - (27) has a unique solution $((q^*_{A0}, q^*_A), (q^*_{B0}, q^*_B))$ for all $(p, y, \omega) \in U \times (0, 1)$, and the function

$$g^*(p, y, \omega(p, y, z)) \quad = \quad (q^*_0, q^*_A + q^*_B) \tag{28}$$

is continuously differentiable in (p, y, ω) on $U \times V$, then the tuple $(\mathcal{I}, u_A, u_B, \omega)$ is called a *Cournot model with one public good.* $\triangle$

The system (26) - (27) defines the Nash equilibrium of a simultaneous-move game in which A and B make voluntary contributions to a public good. Their strategies (and actions) are their contributions q_{A0} and q_{B0}. Their strategy spaces are $[0, y\omega/p_0]$ for A and $[0, y(1-\omega)/p_0]$ for B.[3]

3.2 Scope and Interpretation of the Models

Both collective and Cournot models concern allocation within an existing family. Both the Pareto weight function μ and the relative wealth function ω are atheoretic ways of incorporating "outside influences" into the within-family decision problem. A tempting interpretation of the Pareto weight μ is that it represents the opportunity cost of marriage, but without a specification of what exactly the alternatives to marriage are, it is just a black box that picks out a point on the Pareto frontier.

Cournot models are also silent on the mechanism by which prices and wealth, (p, y), and social circumstances ("distribution factors") z affect allocation within a family. The function ω simply associates each configuration of (p, y, z) with a share of wealth for A. It is possible that the map ω represents equilibrium transfers on the marriage market. This would be the

[3]It is important to assume that there is only one public good, because that assumption, along with a mild normality condition on the preferences of A and B, will be sufficient to guarantee that the equilibrium allocation is unique. I discuss those conditions further in Section 3.3.1.

case if the distribution factors z indicate conditions on the marriage market and help to determine transfers - dowries or bride prices, for example - made between A and B. However, this interpretation is not necessary for the identification arguments I will make.

The major difference between these two classes of models would seem to be that $\omega(p, y, z)$ is at least potentially observable, given a well-designed survey. In practice, it may be very difficult or expensive to gather accurate information about ω. But if a survey were so generously funded that it became feasible to learn about ω, not much more imagination is required to think that such a survey could also collect information on individual-level consumption. And if consumption can be disaggregated from the household to the individual level, a much more direct route to the identification of inefficiency is possible, as I discuss in Section **??**.

3.3 Properties of Cournot Models

3.3.1 Existence and Uniqueness of Equilibrium

Let $\Gamma(U, V)$ denote the set of all Cournot models defined over (U, V), and write $\gamma = (\mathcal{I}, u_A, u_B, \omega)$ for a single Cournot model. Definition 5 requires that the equilibrium defined by the best-response functions (26) - (27) is unique, but it is not immediate that there are preferences u_A and u_B such that it will be.

The following example exhibits preferences u_A and u_B such that the equilibrium exists and is unique for all $(p, y, \omega) \in U \times (0, 1)$; furthermore, the equilibrium allocations $g^*(p, y, \omega)$ depend smoothly on (p, y, ω), except at two points $\omega_*, \omega^* \in (0, 1)$.

Thus, if the map ω is such that neither ω_* nor ω^* are in its range $\omega(U \times V)$, then we will have found a Cournot model γ meeting Definition 5, and hence we can conclude that the set $\Gamma(U, V)$ is nonempty.

Example 3. As in Example 1, let there be three goods in total: a public good q_0 and two private goods q_1, q_2. Again assume preferences are

$$u_A(q_0, q_{A1}) = a \log(q_0) + (1 - a) \log(q_{A1}) \tag{29}$$

$$u_B(q_0, q_{B1}) = b \log(q_0) + (1 - b) \log(q_{B2}) \tag{30}$$

for some $0 < a, b < 1$. Let y_A, y_B be the wealth levels of the two family members.

If we normalize the p_0, the price of the public good, to unity, A's decision problem is

$$\max_{q_{A0}, q_{A1}} a \log(q_{A0} + q_{B0}) + (1 - a) \log(q_{A1}) \quad \text{s.t} \quad q_{A0} + p_1 q_{A1} \leq y_A \tag{31}$$

given B's contribution q_{B0}. But since the total consumption of the public good is $q_0 = q_{A0} + q_{B0}$, we can rewrite A's problem as

$$\max_{q_0, q_{A1}} a \log(q_0) + (1 - a) \log(q_{A1}) \quad \text{s.t} \quad q_0 + p_1 q_{A1} \leq y\omega + q_{B0} \tag{32}$$

$$q_0 \geq q_{B0} \tag{33}$$

which has the solution

$$q^*_{A0}(p, y_A, q_{B0}) = \begin{cases} a(y_A + q_{B0}) - q_{B0} & \text{if } q_{B0} < a(y_A + q_{B0}) \\ 0 & \text{if } q_{B0} \geq a(y_A + q_{B0}) \end{cases} \quad (34)$$

$$p_1 \cdot q^*_{A1}(p, y_A, q_{B0}) = \begin{cases} (1-a)(y_A + q_{B0}) & \text{if } q_{B0} < a(y_A + q_{B0}) \\ y_A & \text{if } q_{B0} \geq a(y_A + q_{B0}) \end{cases} \quad (35)$$

Similarly, B's best response is

$$q^*_{B0}(p, y_B, q_{A0}) = \begin{cases} b(y_B + q_{A0}) - q_{A0} & \text{if } q_{A0} < b(y_B + q_{A0}) \\ 0 & \text{if } q_{A0} \geq b(y_B + q_{A0}) \end{cases} \quad (36)$$

$$p_2 \cdot q^*_{B2}(p, y_B, q_{A0}) = \begin{cases} (1-b)(y_B + q_{A0}) & \text{if } q_{A0} < b(y_B + q_{A0}) \\ y_B & \text{if } q_{A0} \geq b(y_B + q_{A0}) \end{cases} \quad (37)$$

An equilibrium for this "Cournot" game is a pair $(q^*_{A0}, q^*_{B0}) \in [0, y_A/p_0] \times [0, y_B/p_0]$ such that solves (34) and (36) simultaneously; the equilibrium levels of private consumption q^*_{A1} and q^*_{B2} are then implicitly determined by A and B's individual budget constraints.

Let $\omega = y_A/(y_A + y_B)$ be A's share of the household's total wealth, $y = y_A + y_B$. The unique equilibrium of this public-goods game is

$$q^*_{A0} = \begin{cases} 0 & \text{if } \omega < \omega_* \\ y \cdot [a\omega - (1-a)b(1-\omega)] & \text{if } \omega \in (\omega_*, \omega^*) \\ a\omega \cdot y & \text{if } \omega \geq \omega^* \end{cases} \quad (38)$$

$$q^*_{B0} = \begin{cases} y \cdot b(1-\omega) & \text{if } \omega < \omega_* \\ y \cdot [b(1-\omega) - (1-b)a\omega] & \text{if } \omega \in (\omega_*, \omega^*) \\ 0 & \text{if } \omega \geq \omega^* \end{cases} \quad (39)$$

where ω_* and ω^* solve

$$a\omega_* - (1-a)b(1-\omega_*) = 0 \quad (40)$$

$$b\omega^* - (1-b)a\omega^* = 0 \quad (41)$$

i.e.

$$\omega_* = \frac{b(1-a)}{1-(1-a)(1-b)} \quad (42)$$

$$\omega^* = \frac{b}{1-(1-a)(1-b)} \quad (43)$$

Hence, aggregate demands are

$$g_0^*(p,y,\omega) = \begin{cases} y\cdot b(1-\omega) & \text{if } \omega < \omega_* \\ y\cdot \frac{ab}{1-(1-a)(1-b)} & \text{if } \omega \in (\omega_*, \omega^*) \\ y\cdot a\omega & \text{if } \omega \geq \omega^* \end{cases} \tag{44}$$

$$g_1^*(p,y,\omega) = \begin{cases} \frac{y}{p_1}\cdot \omega & \text{if } \omega < \omega_* \\ \frac{y}{p_1}\cdot \frac{(1-a)b}{1-(1-a)(1-b)} & \text{if } \omega \in (\omega_*, \omega^*) \\ \frac{y}{p_1}\cdot (1-a)\omega & \text{if } \omega \geq \omega^* \end{cases} \tag{45}$$

$$g_2^*(p,y,\omega) = \begin{cases} \frac{y}{p_2}\cdot (1-b)(1-\omega) & \text{if } \omega < \omega_* \\ \frac{y}{p_2}\cdot \frac{b}{1-(1-a)(1-b)} & \text{if } \omega \in (\omega_*, \omega^*) \\ \frac{y}{p_2}\cdot (1-\omega) & \text{if } \omega \geq \omega^* \end{cases} \tag{46}$$

Figure 1 depicts the aggregate demands $g^*(p,y,\omega)$ as a function of A's relative wealth, ω. Now, if the range $\omega(\{(p,y)\}\times V)$ is contained in either $(0,\omega_*)$ or $(\omega^*,1)$, then the tuple $\gamma = (\{0\}, u_A, u_B, \omega)$ is a Cournot model in the sense of Definition 5; that is, $\gamma \in \Gamma(U,V)$. $\triangle$

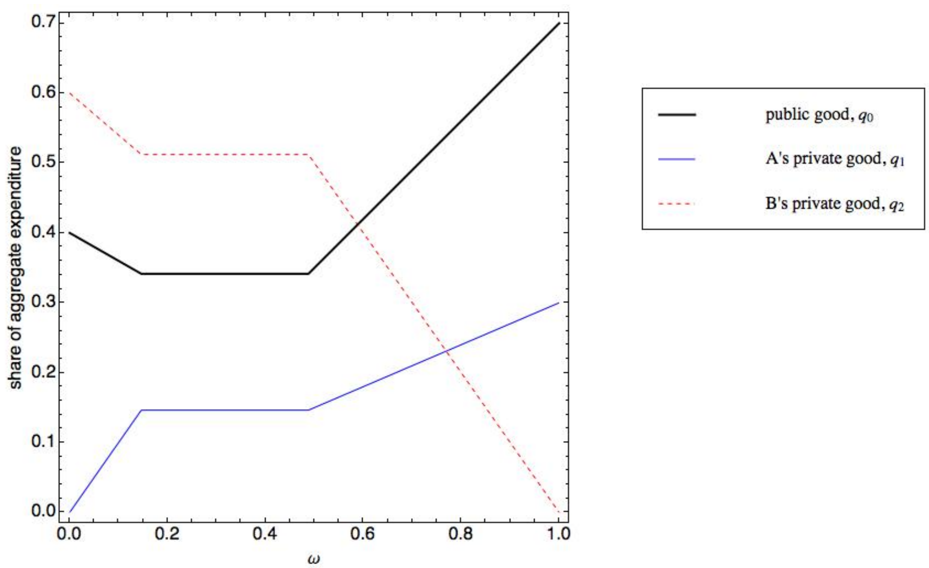

Figure 1: Equilibrium allocations for the Cournot game, $g^*(p, y, \omega)$ under Cobb-Douglas preferences. Share parameters are $a = 0.7$ and $b = 0.4$. The relative prices of the private goods are $p_1 = p_2 = 1$, and total household wealth is $y = 1$.

Example 3 contains two lessons. First, the conditions under which a unique equilibrium of the public goods game exists are fairly mild. In the above example, a unique equilibrium exists whenever the best-response functions (34) and (36) have a unique intersection (q^*_{A0}, q^*_{B0}), and this was always the case because both a and b lay in the interval $(0, 1)$. The fact that preferences were Cobb-Douglas is not essential: as long as the slope of each partners' Engel curve for the public good is strictly between zero and one, there will be a unique equilibrium. In other words, we have the following:

PROPOSITION 1. *Let $q_{A0}(p, y_A)$ be A's Marshallian demand for the public good, and similarly for $q_{B0}(p, y_B)$. If, for all (p, y_A, y_B), $0 < \partial q_{A0}(p, y_A)/\partial y_A < 1$, and $0 < \partial q_{B0}(p, y_B)/\partial y_B < 1$, then the public goods game of Definition 5 has a unique Nash equilibrium.*

Proof. This is simply a restatement of Theorems 2 and 3 of [Bergstrom *et al.* , 1986]). □

Second, by inspection of (44) - (46), one can see that variations in (p, y) do not change the qualitative properties of the aggregate demands: when A is sufficiently poor relative to B, A will not contribute to the public good; when B is sufficiently poor relative to A, the reverse happens; and when A and B are nearly equally endowed, both will contribute to the public good.[4] Formalizing this, we have:

PROPOSITION 2. *Let $q_{A0}(p, y_A)$ be A's Marshallian demand for the public good, and similarly let $q_{B0}(p, y_B)$ be B's Marshallian demand for the public good. Normalize the price of the public good p_0 to unity, and fix the other prices $(p_1 \ldots p_m)$ and the aggregate endowment y. Let A's share of wealth be ω, and write $q^*_A(\omega|p, y)$ and $q^*_B(1-\omega|p, y)$ for the equilibrium quantities of A and B's private consumption. Then there exist scalars $\omega_*(p, y)$, $\omega^*(p, y)$, with $0 < \omega_* < \omega^* < 1$, such that the shares of aggregate consumption devoted to each partner's private consumption are:*

$$y^{-1}\sum_{i=1}^{m} p_i q^*_{Ai}(\omega|p, y) = \begin{cases} \omega & \text{if } \omega \in [0, \omega_*] \\ \omega_* & \text{if } \omega \in (\omega_*, \omega^*) \\ \omega - y^{-1}q_{A0}(p, y\omega) & \text{if } \omega \in [\omega^*, 1] \end{cases} \tag{47}$$

$$y^{-1}\sum_{i=1}^{m} p_i q^*_{Bi}(\omega|p, y) = \begin{cases} (1-\omega) - y^{-1}q_{B0}(p, (1-\omega)y) & \text{if } \omega \in [0, \omega_*] \\ 1 - \omega^* & \text{if } \omega \in (\omega_*, \omega^*) \\ (1-\omega) & \text{if } \omega \in [\omega^*, 1] \end{cases} \tag{48}$$

Proof. See Appendix A. □

[4]It can be shown that when both partners contribute to the public good, its aggregate consumption is invariant to small redistributions of the endowment. (That result is stated in Theorem 1 of [Bergstrom *et al.* , 1986].) But, given the budget constraints of each partner, the value of the private consumption of each partner must be locally constant in ω too, although the composition of their private consumption may change. This is visually obvious in Figure 1, since all budget shares $p_i g^*_i(p, y, \omega)/y$ are flat whenever $\omega \in (\omega_*, \omega^*)$. However, this "income-pooling" zone will not be useful for my analysis, so I leave it aside.

3.3.2 Inefficiency of Equilibrium

Except in extreme cases, such as when the public good and private goods are perfect complements, or when one partner has no wealth, Cournot equilibria are inefficient.

Figure 2 depicts, for fixed (p, y), the set of efficient and equilibrium allocations for a household with the same profile of preferences as in Example 3. The set of efficient allocations is the image of the unit interval $[0, 1]$ under the map $\mu \mapsto g^{**}(p, y, \mu)$. The set of equilibrium allocations, though, is the image of the unit interval under the map $\omega \mapsto g^{*}(p, y, \omega)$. Clearly, the two sets are different, so Cournot equilibria are inefficient.

Just how inefficient the Cournot equilibria are depends on the preferences of the two partners and on the distribution of wealth ω. In Appendix B, I give an expression for the magnitude of the inefficiency when both partners have constant-elasticity of substitution (CES) preferences.

To see why noncooperative equilibria are inefficient, suppose the preferences of the agents are such that the marginal rates of substitution are well-defined and depend smoothly on consumption. An allocation (q_0, q_A, q_B) is efficient only if, for each private good $1 \leq i \leq m$,

$$\frac{p_0}{p_i} \geq \frac{\partial u_A / \partial q_0}{\partial u_A / \partial q_{Ai}}(q_0, q_A) + \frac{\partial u_B / \partial q_0}{\partial u_B / \partial q_{Bi}}(q_0, q_B) \tag{49}$$

with equality if $q_0 > 0$.

The left-hand side of (49) is the relative price of the public good, while the right-hand side is the marginal *social* willingness to pay for the public good. Efficiency requires their equality. In a Cournot equilibrium, though, each person who contributes will have a marginal willingness to pay equal to the relative price; and even those who do not contribute will have a nonzero willingness to pay. The household's social willingness to pay for the public good will therefore be higher than its relative price. Taken the other way around, this means the equilibrium quantity of the public good will be too low.

Having constructed both an efficient and an inefficient model, we are finally in a position to see whether the sort of data described in Section 2 above can distinguish between them.

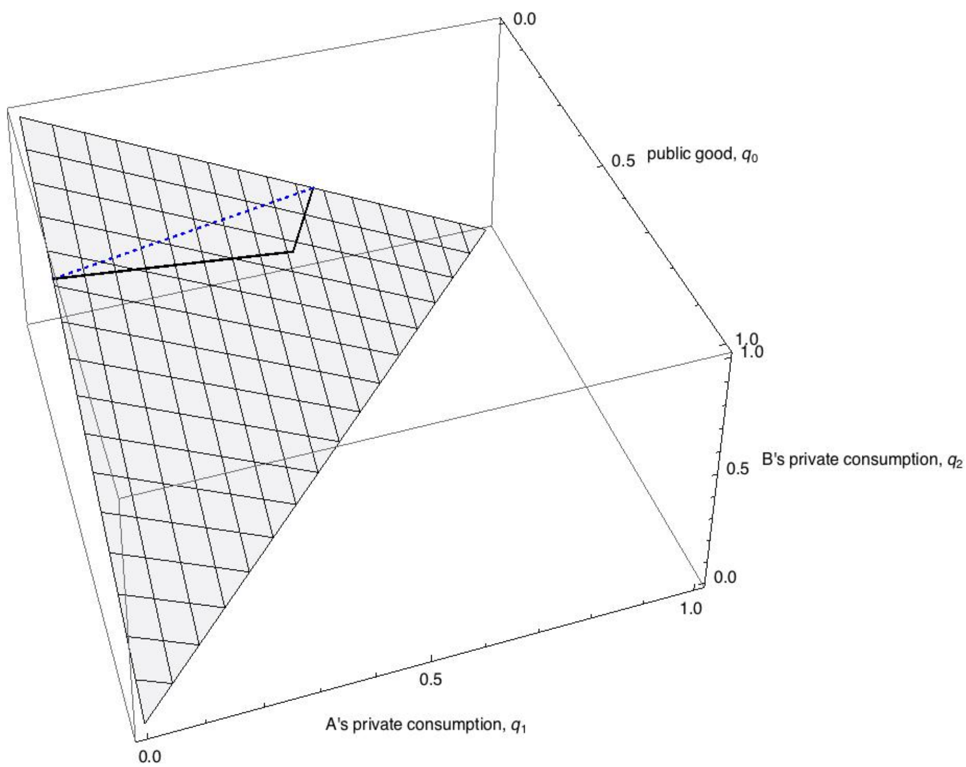

Figure 2: Equilibrium and efficient allocations under Cobb-Douglas preferences. Share parameters are $a = 0.7$ and $b = 0.4$. The relative prices of the private goods are $p_1 = p_2 = 1$, and total household wealth is $y = 1$. The black line is the set of Cournot equilibrium allocations, $g^*(p, y, \omega)$. The dashed blue line is the set of efficient allocations, $g^{**}(p, y, \mu)$.

4 Tests of Distribution Factor Proportionality

By definition, both collective and Cournot models induce extended demand systems on $U \times V$. In view of that fact, I will abuse notation slightly and write $g^{**}(p, y, z|\theta) \in \mathcal{G}(U, V)$ and $g^{*}(p, y, z|\gamma) \in \mathcal{G}(U, V)$. Similarly, at any suitable $\overline{p}$, define $h^{**}(y, z|\theta) \in \mathcal{H}(U, V|\overline{p})$ to be the restricted aggregate demand system implied by the collective model θ at $\overline{p}$:

$$h^{**}(y, z|\theta) \quad = \quad \begin{bmatrix} \overline{p}_0 g_0^{**}(\overline{p}, y, z|\theta) \\ \overline{p}_1 g_1^{**}(\overline{p}, y, z|\theta) \\ \vdots \\ \overline{p}_m g_m^{**}(\overline{p}, y, z|\theta) \end{bmatrix} \tag{50}$$

The obvious analogue for Cournot models is

$$h^{*}(y, z|\gamma) \quad = \quad \begin{bmatrix} \overline{p}_0 g_0^{*}(\overline{p}, y, z|\gamma) \\ \overline{p}_1 g_1^{*}(\overline{p}, y, z|\gamma) \\ \vdots \\ \overline{p}_m g_m^{*}(\overline{p}, y, z|\gamma) \end{bmatrix}. \tag{51}$$

The problem facing the econometrician is to determine whether a given $h \in \mathcal{H}(U, V|\overline{p})$ was generated by a collective model, by a Cournot model, or neither. That is, is $h = h^{**}(\cdot|\theta)$ for some $\theta \in \Theta(U, V)$? Alternatively, is $h = h^{*}(\cdot|\gamma)$ for some $\gamma \in \Gamma(U, V)$? An obvious place to start would be finding conditions on h implied by efficiency. In fact, there are such conditions, as first introduced by [Browning & Chiappori, 1998].

Definition 6. Let $h \in \mathcal{H}(U, V|\overline{p})$ be a restricted aggregate demand system such that no distribution factor is redundant, i.e. for all goods i and all distribution factors z_k,

$$\frac{\partial h_i}{\partial z_k}(y, z) \quad \neq \quad 0 \tag{52}$$

for all $(y, z) \in (U \times V)|_{\overline{p}}$. If, for all $(y, z) \in (U \times V)|_{\overline{p}}$, and all goods i, i' and all distribution factors k, k',

$$\frac{\partial h_i/\partial z_k}{\partial h_i/\partial z_{k'}} = \frac{\partial h_{i'}/\partial z_k}{\partial h_{i'}/\partial z_{k'}} \tag{53}$$

then say that h satisfies *distribution factor proportionality (DFP) on* $(U \times V)|_{\overline{p}}$. $\triangle$

Because the distribution factors only act on demands through the scalar index μ, the chain rule implies that the effects of different distribution factors must be proportional. For if h is induced by a collective model, we can write

$$h(y, z) \quad = \quad h^{**}(y, \mu(y, z)|\theta) \tag{54}$$

Differentiating, we obtain that for any good i, and any two distribution factors k, k',

$$\frac{\partial h_i}{\partial z_k} \quad = \quad \frac{\partial h_i^{**}}{\partial \mu} \cdot \frac{\partial \mu}{\partial z_k} \tag{55}$$

$$\frac{\partial h_i}{\partial z_{k'}} \quad = \quad \frac{\partial h_i^{**}}{\partial \mu} \cdot \frac{\partial \mu}{\partial z_{k'}} \tag{56}$$

Because no distribution factor is redundant, we can divide (55) by (56), so for all goods i,

$$\frac{\partial h_i/\partial z_k}{\partial h_i/\partial z_{k'}} = \frac{\partial \mu/\partial z_k}{\partial \mu/\partial z_{k'}}. \tag{57}$$

In short, we have the following:

Proposition 3. *If the restricted aggregate demand system $h \in \mathcal{H}(U,V|\overline{p})$ is induced by a collective model, h satisfies distribution factor proportionality.*

Proposition 3 appears to give a way of testing efficiency, even with quite limited data. In fact, [Bourguignon *et al.* , 2009] endorse the testing of distribution factor proportionality as the sole implication of efficiency in data without variation in relative prices.[5]

However, distribution factor proportionality is also a property of Cournot models. Just like collective models, Cournot models imply that the distribution factors only act on demands through the scalar index ω. So for the same reasons that collective models imply distribution factor proportionality, Cournot models do too.

Proposition 4. *If the restricted aggregate demand system $h \in \mathcal{H}(U,V|\overline{p})$ is induced by a Cournot model, h satisfies distribution factor proportionality.*

Proof. Let h be induced by a Cournot model. We can write

$$h(y,z) = h^*(y,\omega(y,z)|\gamma) \tag{58}$$

Differentiating and taking ratios as in the proof of Proposition 3, we have the result. □

Proposition 4 means that a test of distribution factor proportionality cannot be interpreted as a test of Pareto efficiency, because it will have no power against Cournot alternatives.

5 Limits to Identifiability

Still, one might hold out hope that the failure of identification highlighted by Proposition 4 might be avoided by testing properties more stringent than distribution factor proportionality.

Unfortunately, I am able show that there are limits to the identifying power of any test, at least when prices do not vary across households. If one uses only information local to a point $(\overline{y},\overline{z}) \in (U \times V)|_{\overline{p}}$ about the conditional mean $h(y,z) = E[Q|P=\overline{p}, Y=y, Z=z]$, then any test that rejects no efficient models will have no power against some inefficient alternatives. Conversely, any test using only local information about demands with power against all Cournot alternatives must reject some efficient models. To see how this type of observational equivalence can arise, I offer the following twist on my running example of a couple with Cobb-Douglas preferences.

[5] When the data do contain variation in relative prices, efficiency also implies a separate set of properties on the price and wealth derivatives of the extended demand system $g^{**}(p,y,\mu(p,y,z))$. For completeness, I state these conditions in Appendix C.

EXAMPLE 4. Suppose that, as in Example 2, the relationship between A's true relative wealth ω and the distribution factors z_A and z_B is

$$\omega(y, z_A, z_B) = \frac{1}{2y}(y + z_A - z_B). \tag{59}$$

Also suppose that, under an efficient model,

$$\mu(y, z_A, z_B) = \frac{1}{2y}(y + z_A - z_B). \tag{60}$$

A's Pareto weight is positively related to her earnings z_A, perhaps because they are correlated with her outside options on the marriage market. Now reconsider the collective model of Example 1. The restricted aggregate demand system in that case was

$$h_0^{**}(y, \mu(y, z_A, z_B)) = \alpha \cdot \frac{1}{2}(y + z_A - z_B) + \beta \cdot \frac{1}{2}(y + z_B - z_A) \tag{61}$$

$$h_1^{**}(y, \mu(y, z_A, z_B)) = \frac{(1-\alpha)}{2} \cdot (y + z_A - z_B) \tag{62}$$

$$h_2^{**}(y, \mu(y, z_A, z_B)) = \frac{(1-\beta)}{2} \cdot (y + z_B - z_A). \tag{63}$$

But in Example 3, we have an example of a Cournot model that results, at least over a certain part of its domain, in the restricted aggregate demands

$$h_0^{*}(y, \omega(y, z_A, z_B)) = a \cdot \frac{1}{2}(y + z_A - z_B) \tag{64}$$

$$h_1^{*}(y, \omega(y, z_A, z_B)) = (1-a) \cdot \frac{1}{2}(y + z_A - z_B) \tag{65}$$

$$h_2^{*}(y, \omega(y, z_A, z_B)) = \frac{1}{2}(y + z_B - z_A). \tag{66}$$

A comparison of (64)-(66) with (61) - (63) reveals that if $\alpha = a$ and $\beta = 0$, the aggregate demands implied by the Cournot model and those implied by the collective model are exactly the same, for all (y, z). So the set of allocations generated by the efficient and the inefficient models coincide exactly. Yet one is Pareto efficient, and the other is not.

$\triangle$

The economics of Example 4 are easy to state: without knowing who is paying for what, or the preferences of the two agents, it is impossible to tell if B is free-riding by withholding contributions to the public good q_0, as in the Cournot model, or if B simply does not like the public good, as in the collective model.[6] Put differently, the solution concept and the

[6] The observational equivalence of the two models can be interpreted geometrically, too. In Figure 2, the set of equilibrium allocations (the solid black line) is distinct from the set of efficient allocations (the dashed blue line) for a fixed profile of preferences (u_A, u_B), summarized here by the parameters (a, b). However, the particular location and slope of the set of equilibrium allocations depends on preferences. So for a different set of preferences - say (α, β) - the set of equilibrium allocations for (a, b) and the set of efficient allocations for (α, β) can overlap, at least partly. If the data are such that only allocations in the overlap are observed, it will be impossible to distinguish between efficiency and inefficiency.

preferences of the household members need to be identified jointly. Efficiency is a concept that only makes sense with respect to a fixed set of preferences and production possibilities; one cannot claim that outcomes are efficient but then refuse to say what "efficiency" means.

Proposition 5 below generalizes Example 4 by exhibiting a class of demand systems that are locally consistent with both collective and Cournot models. Roughly speaking, a restricted aggregate demand system h is consistent with a collective model over a given domain if there is a collective model θ such that $h(y,z) = h^{**}(y,z|\theta)$ identically.

I need to work with the weaker concept of a demand system being "consistent with" a given model, rather than being "generated by" that model, because with data observed over a bounded domain, and at only one set of relative prices $\overline{p}$, it will obviously be impossible to learn about the preferences of the two household members over bundles that are never affordable. Definition 7 below states this idea formally.

Definition 7. Let $h \in \mathcal{H}(U,V|\overline{p})$ be a restricted aggregate demand system and let $\mathcal{I} \subseteq \{0,1,\ldots m\}$ be a set with $0 \leq n \leq m+1$ elements. By permuting indices, we may assume $\mathcal{I} = \{0,1,\ldots n-1\}$. Let $D_{AB} \subset \mathbb{R}^n_{++}$ be open and convex, and similarly let $D_A, D_B \subset \mathbb{R}^{m+1-n}_{++}$ be open and convex. Suppose that for all $(y,z) \in (U \times V)|_{\overline{p}}$,

$$h(y,z) \quad \in \quad D_{AB} \times (D_A + D_B) \tag{67}$$

where $D_A + D_B$ is the Minkowski sum of D_A and D_B. Suppose also that there are two increasing and quasiconcave functions $u_A : D_{AB} \times D_A \longrightarrow \mathbb{R}$ and $u_B : D_{AB} \times D_B \longrightarrow \mathbb{R}$, and a smooth function $\mu : (U \times V)|_{\overline{p}} \longrightarrow (0,1)$ such that

$$\begin{aligned} h(y,z) \quad &= \quad \arg \max_{(q_{AB},q_A,q_B)} \mu \cdot u_A(q_{AB},q_A) + [1-\mu] \cdot u_B(q_{AB},q_B) \\ &\text{subject to} \quad \sum_{i=0}^{n-1} p_i q_{AB,i} + \sum_{i=n}^{m} p_i (q_{A,i} + q_{B,i}) \leq y \end{aligned} \tag{68}$$

for all $(y,z) \in (U \times V)|_{\overline{p}}$. Then we say that h *is consistent with a collective model with* n *public goods over* $(U \times V)|_{\overline{p}}$. $\triangle$

Similarly, to say that a given restricted aggregate demand system h is consistent with a Cournot model is to say that there is a Cournot model γ such that, over its domain, $h(y,z) = h^*(y,z|\gamma)$ identically.

Definition 8. Let $h \in \mathcal{H}(U,V|\overline{p})$ be a restricted aggregate demand system and let $\mathcal{I} \subseteq \{0,1,\ldots m\}$ be a singleton. By permuting indices, we may assume $\mathcal{I} = \{0\}$. Let D_{AB} be an open interval, and let $D_A, D_B \subset \mathbb{R}^m_{++}$ be open and convex. Suppose that for all $(y,z) \in (U \times V)|_{\overline{p}}$,

$$h(y,z) \quad \in \quad D_{AB} \times (D_A + D_B) \tag{69}$$

where $D_A + D_B$ is the Minkowski sum of D_A and D_B. Suppose also that there are two increasing and quasiconcave functions $u_A : D_{AB} \times D_A \longrightarrow \mathbb{R}$ and $u_B : D_{AB} \times D_B \longrightarrow \mathbb{R}$, and a smooth function $\omega : (U \times V)|_{\overline{p}} \longrightarrow (0,1)$ such that the system

$$\max_{(q_{A0},q_A)} u_A(q_{A0}+q_{B0},q_A) \quad \text{s.t.} \quad p_0 q_{A0} + \sum_{i=1}^{m} p_i q_{Ai} \leq y\omega \tag{70}$$

$$\max_{(q_{B0},q_B)} u_B(q_{A0}+q_{B0},q_B) \quad \text{s.t} \quad p_0 q_{B0} + \sum_{i=1}^{m} p_i q_{Bi} \leq y(1-\omega) \tag{71}$$

has a unique solution $((q^*_{A0}, q^*_A), (q^*_{B0}, q^*_B))$ for all $(y,z) \in (U \times V)|_{\overline{p}}$, and further that

$$h_0(y,z) = q^*_{A0} + q^*_{B0} \tag{72}$$

$$h_i(y,z) = q^*_{Ai} + q^*_{Bi} \text{ for all } i = 1, \ldots m. \tag{73}$$

Then we say h *is consistent with a Cournot model over* $(U \times V)|_{\overline{p}}$. △

Define, for each possible domain $(U \times V)|_{\overline{p}}$, the family of demand systems

$$\mathcal{H}_0^{(n)}(U,V|\overline{p}) = \{h \in \mathcal{H}(U,V|\overline{p}) : h \text{ is consistent with a collective model with } n \text{ public goods}\}. \tag{74}$$

Their union is the set of demand systems which are consistent with at least one collective model:

$$\mathcal{H}_0(U,V|\overline{p}) = \bigcup_{n=0}^{m+1} \mathcal{H}_0^{(n)}(U,V|\overline{p}). \tag{75}$$

Also define the set of demand systems which are consistent with at least one Cournot model:

$$\mathcal{H}_1(U,V|\overline{p}) = \{h \in \mathcal{H}(U,V|\overline{p}) : h \text{ is consistent with a Cournot model}\}. \tag{76}$$

I will show that $\mathcal{H}_0(U,V|\overline{p})$ and $\mathcal{H}_1(U,V|\overline{p})$ are not well-separated, in the following sense:

PROPOSITION 5. *Let $(\overline{y},\overline{z}) \in (U \times V)|_{\overline{p}}$ be given. There is an open neighborhood $U' \times V' \subset U \times V$ of $(\overline{p},\overline{y},\overline{z})$ such that*

$$\mathcal{H}_0(U',V'|\overline{p}) \cap \mathcal{H}_1(U',V'|\overline{p}) \neq \emptyset. \tag{77}$$

I establish Proposition 5 by showing directly that there is a nonempty set of restricted aggregate demand systems that belongs both to $\mathcal{H}_0(U,V|\overline{p})$ and to $\mathcal{H}_1(U,V|\overline{p})$, at least when the domain $(U \times V)|_{\overline{p}}$ is restricted to a (perhaps small) neighborhood of a given point $(\overline{y},\overline{z})$.

In fact, it will be easy to describe some of the demand systems in the intersection. Consider restricted aggregate demand systems that are *linear*, i.e. ones of the form

$$h(y,z) = \begin{bmatrix} h_{00} & h_{01} & \ldots & h_{0K} \\ h_{10} & h_{11} & \ldots & h_{1K} \\ \vdots & & \ddots & \\ h_{m0} & h_{m1} & \ldots & h_{mK} \end{bmatrix} \times \begin{bmatrix} y \\ z_1 \\ \vdots \\ z_K \end{bmatrix}. \tag{78}$$

Let $\mathcal{H}^L(U,V|\overline{p}) \subset \mathcal{H}(U,V|\overline{p})$ be the set of all such h. Let $\mathcal{H}^{ND}(U,V|\overline{p})$ be the set of all nondegenerate h, in the sense of Definition 3. Lastly, let $\mathcal{H}^{DFP}(U,V|\overline{p}) \subset \mathcal{H}(U,V|\overline{p})$ be the set of restricted aggregate demand systems satisfying distribution factor proportionality.

PROPOSITION 6. *Let $(\overline{y}, \overline{z}) \in (U \times V)|_{\overline{p}}$ be given. There is an open neighborhood $U' \times V' \subset U \times V$ of $(\overline{p}, \overline{y}, \overline{z})$ such that if h is linear, nondegenerate, and satisfies distribution factor proportionality, then h is consistent with a Cournot model over $(U' \times V')|_{\overline{p}}$.*

Proof. See Appendix A. □

PROPOSITION 7. *Let $(\overline{y}, \overline{z}) \in (U \times V)|_{\overline{p}}$ be given. There is an open neighborhood $U' \times V' \subset U \times V$ of $(\overline{p}, \overline{y}, \overline{z})$ such that if h is linear, nondegenerate, and satisfies distribution factor proportionality, then h is consistent with a collective model over $(U' \times V')|_{\overline{p}}$.*

Proof. See Appendix A. □

Proof of Proposition 5. Proposition 6 shows that

$$\mathcal{H}^L(U', V'|\overline{p}) \cap \mathcal{H}^{ND}(U', V'|\overline{p}) \cap \mathcal{H}^{DFP}(U', V'|\overline{p}) \quad \subset \quad \mathcal{H}_1(U, V'|\overline{p}). \tag{79}$$

But Proposition 7 shows that

$$\mathcal{H}^L(U', V'|\overline{p}) \cap \mathcal{H}^{ND}(U', V'|\overline{p}) \cap \mathcal{H}^{DFP}(U', V'|\overline{p}) \quad \subset \quad \mathcal{H}_0(U, V'|\overline{p}) \tag{80}$$

too. Examples 1 and 3 provide examples of restricted aggregate demand systems in $\mathcal{H}^L(U, V'|\overline{p}) \cap \mathcal{H}^{ND}(U, V'|\overline{p}) \cap \mathcal{H}^{DFP}(U, V'|\overline{p})$. Thus,

$$\mathcal{H}^L(U', V'|\overline{p}) \cap \mathcal{H}^{ND}(U', V'|\overline{p}) \cap \mathcal{H}^{DFP}(U', V'|\overline{p}) \quad \neq \quad \emptyset. \tag{81}$$

So we have $\mathcal{H}_0(U', V'|\overline{p}) \cap \mathcal{H}_1(U', V'|\overline{p}) \neq \emptyset$, as required. □

Since distribution factors very rarely have a natural scale, arguments for using global information about the relationship between the distribution factors and household consumption patterns to identify inefficiency are bound to be somewhat contrived. So even though Proposition 5 is only local in nature, it still severely constrains the use of variation in distribution factors alone to identify inefficiency.

6 Restoring Identifiability

But inefficiency *is* a well-defined concept, even if its presence is not detectable in some settings. Identification problems arise when the data are too poor, or when the set of models are too rich. The econometrician can escape the negative conclusions of Proposition 5 either by gathering different data, or by restricting the class of models $\Theta(U, V)$ and $\Gamma(U, V)$ he is willing to entertain.

To prove Propositions 4 and 5, I used the assumptions that (a) the data contain no variation in relative prices, (b) only information about aggregate consumption and income is available, and (c) other than weak monotonicity and convexity, both agents' preferences can be be arbitrary. Perhaps surprisingly, there are reasons to think that introducing variation in prices will not prove useful. The binding constraints on inference are more likely aggregation and lack of information about preferences.

6.1 Price Variation

Without variation in relative prices, both collective and Cournot models can only generate variation in consumption through income effects, as either aggregate wealth or its distribution varies. In a public-goods economy, inefficiency arises from substitution away from public consumption and towards private consumption, so data without price variation cannot be informative about the degree of inefficiency. (Consider the extreme case where public and private goods are perfect complements; then there is no inefficiency. But an agent's Engel curves are uninformative about the degree of substitutability, so the same data can be consistent with a strictly positive level of inefficiency too.) As argued in Section 4, tests using this sort of data will have no power against a Cournot alternative. Much of the evidence in Table 2 is therefore not persuasive.

In the presence of price variation, the aggregate demand of an efficient household has to obey restrictions other than distribution factor proportionality. [Browning & Chiappori, 1998] show that under efficiency, the household analogue of the Slutsky substitution matrix has to be decomposable as the sum of a negative semidefinite matrix and a matrix of rank at most one. For completeness, I derive that condition, sometimes known as "SR1", in Appendix C.

Yet Theorem 4 of [Lechene & Preston, 2011] shows that the aggregate demands of a non-cooperative household will obey the same "SR1" conditions as will an efficient household, at least when each partner's preferences for private goods are weakly separable from the public good. So the prospects for the identification of inefficiency from aggregate household consumption patterns, even with price variation, are not good.

6.2 Disaggregated Data

Now suppose, contrary to my assumptions in Section 2, that the econometrician knows y_A and y_B, not just $y = y_A + y_B$. Then he also knows the relative wealth ω of each partner.

If the household's expenditures cannot be classified into public consumption and the private consumption of each member, disaggregated income data would not help in identifying inefficiency, because ω itself can be a distribution factor - a component of z - and the collective model imposes no restrictions on the form of the Pareto weight function μ. Disaggregated income data can help rule out some inefficient models, though.

Cournot models generate relationships of the form $q = h^*(y, \omega(y, z))$. They cannot explain a relationship between distribution factors and consumption, holding relative wealth constant. Finding a relationship between z and consumption conditional on ω would simply re-open the basic question of this paper: is there is a class of inefficient models against which the collective model is identified?

However, distribution factors typically add little explanatory power to models of household consumption, and from an economic point of view, they are only of indirect interest. So it is still reasonable to ask if there are testable implications of efficiency - and efficiency alone - in an environment where the relationship between the intrahousehold distribution of wealth and consumption can be perfectly observed. That is, suppose the demand system $h : W \longrightarrow \mathbb{R}^{m+1}_{++}$ were known over some open domain $W \subset \mathbb{R}_{++} \times (0, 1)$, and satisfied

$$\sum_{i=0}^{m} h_i(y, \omega) \quad = \quad 1 \tag{82}$$

for all $(y, \omega) \in W$. Would it be possible to reject efficiency on the basis of that knowledge? If some information about the distribution of consumption is available, and preferences satisfy a certain asymmetry in income effects, it is indeed possible.

Suppose we can classify some of the household's expenditures as being either public, exclusively consumed by A, or exclusively consumed by B. In particular, suppose that the econometrician knows that good $i = 0$ is public, that good 1 is privately consumed by A alone, and good 2 is privately consumed by B alone. This is not a complete disaggregation of consumption, but it is more information than is typically used in the literature.

Assume that the preferences of both parties for the private goods are separable from the public good, so that u_A and u_B are of the form

$$u_A(q_0, q_A) \quad = \quad u_A(q_0, v_A(q_A)) \tag{83}$$

$$u_B(q_0, q_B) \quad = \quad u_A(q_0, v_B(q_B)) \tag{84}$$

and that the functions v_A and v_B are homogenous of degree one. Also assume that the assignable private goods 1 and 2 satisfy an Inada condition, so

$$\lim_{q_1 \to 0^+} \frac{\partial u_A}{\partial q_{A1}}(q_0, q_A) \quad = \quad +\infty \tag{85}$$

$$\lim_{q_2 \to 0^+} \frac{\partial u_B}{\partial q_{B2}}(q_0, q_B) \quad = \quad +\infty. \tag{86}$$

Further, assume that for all efficient allocations $(q_0^{**}, q_A^{**}, q_B^{**})$,

$$\frac{\partial}{\partial v_A}\left(\frac{\partial u_A/\partial q_0}{\partial u_A/\partial v_A}(q_0^{**}, v_A(q_A^{**}))\right) \quad \neq \quad \frac{\partial}{\partial v_B}\left(\frac{\partial u_B/\partial q_0}{\partial u_B/\partial v_B}(q_0^{**}, v_B(q_B^{**}))\right). \tag{87}$$

Finally, I will restrict the set of collective models by assuming that for all y, the Pareto weight function $\mu(y, \omega)$ is surjective. That is, for any $\widetilde{\mu} \in (0, 1)$, there is some $\widetilde{\omega} \in (0, 1)$ such that $\mu(y, \widetilde{\omega}) = \widetilde{\mu}$.

With these restrictions on preferences and the set of collective models, a large-support condition on ω will be enough to identify inefficiency. Let y be given and recall that the physical units of the goods are chosen such that all prices are unity. Then we have

PROPOSITION 8. *Suppose A and B's preferences for the private goods are separable from the public good, their preferences for the private goods are homogenous of degree one, and their preferences for the assignable goods satisfy the Inada conditions (85)-(86). Suppose also that* *the support of the conditional distribution of* ω *at* y *is the full unit interval:* $\inf\{\omega : (y, \omega) \in W\} = 0$*, and* $\sup\{\omega : (y, \omega) \in W\} = 1$*. Let*

$$\widetilde{\omega} \quad = \quad \arg\min h_0(y, \omega) \tag{88}$$

Then for any collective model such that $\mu(y, \omega)$ *is surjective at* y*,*

$$\min\{h_1(y, \widetilde{\omega}), h_2(y, \widetilde{\omega})\} \quad = \quad 0. \tag{89}$$

Proof. See Appendix A. □

Importantly, the condition (89) is both testable and *not* true of Cournot models. Figure 1 illustrates the logic of this result: in noncooperative households, the lowest level of public consumption occurs when the distribution of income ω is relatively equal, meaning that neither agent's private consumption will be zero. However, in cooperative households, the lowest level of public consumption occurs when the partner who cares least about the public good has all the bargaining power, meaning that one agent's private consumption has to be zero.

Proposition 9. *Suppose A and B's preferences for the private goods are separable from the public good, their preferences for the private goods are homogenous of degree one, and their preferences for the assignable goods satisfy the Inada conditions (85)-(86). Suppose also that the support of the conditional distribution of* ω *at* y *is the full unit interval:* $\inf\{\omega : (y, \omega) \in W\} = 0$*, and* $\sup\{\omega : (y, \omega) \in W\} = 1$*. Let*

$$\widetilde{\omega} \quad = \quad \arg\min h_0(y, \omega) \tag{90}$$

Then for any Cournot model,

$$\min\{h_1(y, \widetilde{\omega}), h_2(y, \widetilde{\omega})\} \quad > \quad 0. \tag{91}$$

Proof. See Appendix A. □

6.3 Preference Restrictions

Relatedly, we could restrict the preferences of the household members. In data without price variation, assumptions about preferences can help fill in missing information about substitution effects. If we were willing to go so far as to fully specify both members' preferences, we could directly compute the efficient and the equilibrium allocations. Testing a collective model against a Cournot one would then reduce to the question of which model provides a better fit to the consumption data.

In fact, the identification problem discussed here is as much the fault of the complete lack of restrictions on preferences as it is the fault of limited data on individual consumption and wealth.

[Chiappori & Ekeland, 2009] show, in a similar setting to mine, that household-level data *with* price variation cannot identify the preferences of a household's members even within the class of collective models. In view of those results, the failure of identification I establish should not be surprising. Without restricting the permissible class of preferences for the household's members beyond some weak nonsatiation and convexity requirements, both efficient and inefficient models of the family are very "high-dimensional" objects, and restricting the information available by considering only data without price variation should only make the identification problem worse.

Table 2: Studies Failing to Reject Efficiency in Household Consumption. Each author estimates either an extended or restricted aggregate demand system and tests the implied restrictions listed under "Properties Tested". I abbreviate "distribution factor proportionality" to "DFP". "SR1" is a property that can only be tested in data with relative price variation, and is derived in Appendix C.

Paper	**Country**	**Goods**	**Distribution Factors Used**	**Properties Tested**
[Attanasio & Lechene, 2014]	Mexico	starches pulses fruit and vegetables meat, fish, and dairy other foods	relative nonlabor income relative size and wealth of extended family	DFP
[Bayudan, 2006]	Philippines	time use (wives only)	relative wages self-reported influence	DFP
[Bobonis, 2009]	Mexico	child clothing men's and women's clothing schooling food categories	relative nonlabor income rainfall	DFP
[Bourguignon *et al.* , 1993]	France	men's and women's clothing food at home food at restaurants health care cosmetics books and music "vacation" entertainment	relative nonlabor income	DFP
[Browning *et al.* , 1994]	Canada	men's and women's clothing	relative age relative labor income	DFP

[Browning & Chiappori, 1998]	Canada	men's and women's clothing food at home food at restaurants transportation "services" alcohol and tobacco	relative age relative labor income	SR1; DFP
[Chiappori *et al.* , 2002]	US	labor supply	local sex ratio local divorce laws	DFP
[Donni, 2007]	France	labor supply (wife only) food (aggregated) clothing recreation transportation	relative wages	DFP
[Donni & Moreau, 2007]	France	labor supply (wife only) food at home	relative wages relative age education of wife	DFP
[Fortin & Lacroix, 1997]	Canada	labor supply	relative nonlabor income relative wages	DFP
[Vermeulen, 2005]	Netherlands	labor supply	relative nonlabor income relative age marital status	DFP

7 Conclusion

None of my arguments have been empirical, so the results presented in this paper do not mean that families are actually inefficient. On theoretical grounds, in fact, one might be skeptical of the idea that inefficiencies can persist in the sort of long-term partnership that is family life. But the inefficiency arising from the underprovision of public goods consists only of *relative* waste. In particular, a Cournot equilibrium is inefficient relative to a first-best allocation within an *existing* marriage. So even in an inefficient equilibrium, each partner is better off than under autarky, and so it can be individually rational to tolerate some free-riding. If the sort of inefficiencies described in this paper persist, the ultimate fault may lie in the marriage market, perhaps due to search costs or other frictions.

Instead of looking for testable implications of intrafamily efficiency, it may be more fruitful to look for the implications of *inefficiency*. If we take the view that the provision of public goods, such as children or housing, is an important economic function of families, then determining whether families are efficient requires more information about preferences for those goods than economists have thus far brought to bear on this question.

A Proofs of Propositions

A.1 Proofs for Section 3

Proof of Proposition 2. Let $\overline{q}_{A0}(y(1-\omega)|p)$ solve

$$\overline{q}_{A0} = q_{B0}(y(1-\omega)+\overline{q}_{A0}) \tag{92}$$

and similarly let $\overline{q}_{B0}(y\omega|p)$ solve

$$\overline{q}_{B0} = q_{A0}(y\omega+\overline{q}_{B0}) \tag{93}$$

If B's contribution to the public good, q_{B0}, is greater than $\overline{q}_{B0}(y\omega|p)$, A will not contribute, and similarly $q_{A0} \geq \overline{q}_{A0}(y(1-\omega)|p)$ implies that B's best response is to free-ride by setting $q_{B0} = 0$.

Define $\omega_*(p, y)$ and $\omega^*(p, y)$ as the unique solutions to

$$q_{B0}((1-\omega_*)y|p) = \overline{q}_{B0}(y\omega|p) \tag{94}$$

$$q_{A0}(\omega^* y|p) = \overline{q}_{A0}(y(1-\omega)|p) \tag{95}$$

Now, if ω is such that A does not contribute, then

$$q_{B0}((1-\omega)y|p) \geq \overline{q}_{B0}(y\omega|p) \tag{96}$$

but since $q_{B0}((1-\omega)y|p)$ is decreasing in ω and $\overline{q}_{B0}(y\omega|p)$ is increasing in ω, we must have $\omega \leq \omega_*$. Similar reasoning implies that B does not contribute - $q_{B0} = 0$ - exactly when A is sufficiently rich: $\omega \geq \omega^*$. Finally, $\omega_* < \omega^*$ because otherwise there would exist $\omega \in [\omega^*, \omega_*]$, but for such an ω, neither A nor B would contribute to the public good, and that cannot be an equilibrium. But this contradicts Theorem 2 of [Bergstrom *et al.* , 1986], which establishes that in this game, an equilibrium always exists. □

A.2 Proofs for Section 5

Proposition 2 shows that in equilibria where only one partner - say A - contributes to the public good, aggregate consumption of the public good is simply q^*_{A0}, and aggregate consumption of all the m private goods is $q^*_{Aj} + q^*_{Bj}$, for all $j \neq 0$. To prove Proposition 6, I reverse-engineer that equilibrium outcome locally by finding a good i - say it is good 0 - and a scalar $a \in (0,1)$ and *defining* $\omega(y,z)$ by the relation $h_0(y,z) = a\omega(y,z)$, so A's Engel curve for the "public" good is linear by construction.

Then, I decompose the consumption of the remaining m goods as

$$h_j(y,z) \quad = \quad q_{Aj}(\omega(y,z)) + q_{Bj}(1-\omega(y,z)) \tag{97}$$

for $2m$ weakly positive, weakly increasing functions q_{Aj} and q_{Bj} such that "adding-up" holds:

$$a\omega(y,z) + \sum_{j=1}^{m} q_{Aj}(\omega(y,z)) \quad = \quad \omega(y,z) \tag{98}$$

$$\sum_{j=1}^{m} q_{Bj}(1-\omega(y,z)) \quad = \quad 1-\omega(y,z) \tag{99}$$

Finally, I construct preferences u_A, u_B such that A and B's Engel curves are exactly q_{Aj} and q_{Bj} for all the "private" goods $1 \leq j \leq m$, and such that B does not contribute in equilibrium. This is always possible, I show, if B's preference for the public good is weak enough.

Lemma 1. *Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^{m+1}$ be given by*

$$h(y,s) \quad = \quad \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \\ \vdots & \\ h_{m0} & h_{m1} \end{bmatrix} \times \begin{bmatrix} y \\ s \end{bmatrix}. \tag{100}$$

Suppose also that for all i, $h_{i0} > 0$, $h_{i1} \neq 0$, and

$$1 \quad = \quad \sum_{i=0}^{m} h_{i0} \tag{101}$$

$$0 \quad = \quad \sum_{i=0}^{m} h_{i1}. \tag{102}$$

Then there is an integer i, $0 \leq i \leq m$, a scalar $a \in (0,1)$, and a smooth function $\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that (by permuting indices such that $i = 0$),

$$h(y,s) \quad = \quad \begin{bmatrix} a & 0 \\ \delta^A_1 & \delta^B_1 \\ \vdots & \\ \delta^A_m & \delta^B_m \end{bmatrix} \times \begin{bmatrix} y\omega(y,s) \\ y(1-\omega(y,s)) \end{bmatrix} \tag{103}$$

for all $(y,s) \in \mathbb{R}^2$*. Furthermore, the constants* δ_j^A, δ_j^B *can be chosen to be nonnegative and such that*

$$\sum_{j\neq i}^{m} \delta_j^A = 1 - a \tag{104}$$

$$\sum_{j\neq i}^{m} \delta_j^B = 1. \tag{105}$$

Proof of Lemma 1. Since all h_{j1} are nonzero, partition $\{0, 1 \ldots m\}$ into

$$\mathcal{J}^+ = \{j : h_{j1} > 0\} \tag{106}$$

$$\mathcal{J}^- = \{j : h_{j1} < 0\}. \tag{107}$$

Both $\mathcal{J}^+$ and $\mathcal{J}^-$ are nonempty, because $\sum_{j=0}^{m} h_{j1} = 0$. Let

$$i \in \arg\min \left\{ \frac{h_{j0}}{h_{j1}} : j \in \mathcal{J}^+ \right\}. \tag{108}$$

and define, for all $j \neq i$,

$$\delta_j^B = h_{j0} - h_{i0}\frac{h_{j1}}{h_{i1}} \tag{109}$$

$$\begin{aligned} \delta_j^A &= \delta_j^B + a\frac{h_{j1}}{h_{i1}} \\ &= h_{j0} + (a - h_{i0})\frac{h_{j1}}{h_{i1}} \end{aligned} \tag{110}$$

for an $a \in (0, 1)$ that is, for now, unspecified. By construction, $\delta_j^B \geq 0$ for all $j \in \mathcal{J}^+$, and clearly $\delta_j^B \geq 0$ for $j \in \mathcal{J}^-$. This implies that if $j \in \mathcal{J}^+$, then $\delta_j^A \geq 0$ whenever $a \geq 0$. Now, suppose that for all $j \in \mathcal{J}^-$

$$\begin{aligned} a &\leq h_{i0} - h_{j0}\frac{h_{i1}}{h_{j1}} \\ &= h_{i0} + h_{j0}\left|\frac{h_{i1}}{h_{j1}}\right| \end{aligned} \tag{111}$$

If so, then for all $j \in \mathcal{J}^-$,

$$\begin{aligned} \delta_j^A &= h_{j0} + (a - h_{i0})\frac{h_{j1}}{h_{i1}} \\ &= h_{j0} - (a - h_{i0})\left|\frac{h_{j1}}{h_{i1}}\right| \geq 0. \end{aligned} \tag{112}$$

So, let a be such that

$$0 < a < \min\left\{1, h_{i0} + h_{i1} \cdot \min_{j\in\mathcal{J}^-}\left|\frac{h_{j0}}{h_{j1}}\right|\right\} \tag{113}$$

Define

$$\omega(y,s) \quad = \quad \frac{1}{ay} h_i(y,s). \tag{114}$$

For any $j \neq i$,

$$\begin{aligned} h_j(y,s) &= y \cdot h_{j0} + s \cdot h_{j1} \\ &= y \cdot h_{j0} + h_{j1} \cdot \frac{1}{h_{i1}} \left[ay\omega(y,s) - y \cdot h_{i0}\right] \\ &= y \cdot \left\{ \left[h_{j0} - h_i 0 \cdot \frac{h_{j1}}{h_{i1}}\right] + \left[a \frac{h_{j1}}{h_{i1}}\right] \omega(y,s) \right\} \\ &= y \cdot \left\{ \left[h_{j0} - h_i 0 \cdot \frac{h_{j1}}{h_{i1}}\right] \cdot \left[(1 - \omega(y,s)) + \omega(y,s)\right] + \left[a \frac{h_{j1}}{h_{i1}}\right] \omega(y,s) \right\} \\ &= \delta_j^A \cdot y\omega(y,s) + \delta_j^B \cdot y(1 - \omega(y,s)). \end{aligned} \tag{115}$$

□

COROLLARY 2. *Let $h(y,s)$ be as in Lemma 1 above. If $\overline{y} > 0$, a can be chosen such that $\omega(\overline{y},0) \in (0,1)$.*

Proof. If we choose a such that

$$0 < h_{i0} < \quad a \quad < \min\left\{1, h_{i0} + h_{i1} \cdot \min_{j \in \mathcal{J}^-} \left|\frac{h_{j0}}{h_{j1}}\right|\right\} \tag{116}$$

then

$$\begin{aligned} \omega(\overline{y},0) &= \frac{1}{a\overline{y}} h_i(\overline{y},0) \\ &= \frac{h_{i0}}{a} < 1. \end{aligned} \tag{117}$$

□

Proof of Proposition 6. Let $h \in \mathcal{H}^L(U,V|\overline{p}) \cap \mathcal{H}^{ND}(U,V|\overline{p}) \cap \mathcal{H}^{DFP}(U,V|\overline{p})$ be given by

$$h(y,z) \quad = \quad \begin{bmatrix} h_{00} & h_{01} & \dots & h_{0K} \\ h_{10} & h_{11} & \dots & h_{1K} \\ \vdots & & \ddots & \\ h_{m0} & h_{m1} & \dots & h_{mK} \end{bmatrix} \times \begin{bmatrix} y \\ z_1 \\ \vdots \\ z_K \end{bmatrix}. \tag{118}$$

Because h is nondegenerate, $h_{ik} \neq 0$ for all goods i. And because distribution factors have no natural scale, is without loss of *economic* generality to assume $\overline{z} = 0$. h satisfies adding-up, so for any y,

$$\begin{aligned} \sum_{i=0}^{m} h_i(y,\overline{z}) &= \left(\sum_{i=0}^{m} h_{i0}\right) y \\ &= y \end{aligned} \tag{119}$$

which means $\sum_{i=1}^m h_{i0} = 1$. Then because budget shares are strictly positive, $h_i(\overline{y}, \overline{z})/\overline{y} = h_{i0} > 0$. But again by adding up,

$$\begin{aligned}\sum_{i=0}^{m} h_i(0,z) &= \sum_{i=1}^{m}\sum_{k=1}^{K} h_{ik} z_k \\ &= 0. \end{aligned} \tag{120}$$

By considering $z = (1, 0, \ldots 0)$, we see that $\sum_{i=0}^m h_{i1} = 0$.

Next, let

$$s(z) = \sum_{k=1}^{K} \frac{h_{ik}}{h_{i1}} z_k. \tag{121}$$

Because h satisfies distribution factor proportionality, the ratio h_{ik}/h_{i1} does not depend on the choice of good i, so s is well-defined.

For all goods i, and all $(y, z) \in (U \times V)|_{\overline{p}}$, then,

$$\begin{aligned}h_i(y,z) &= y \cdot h_{i0} + h_{i1} \sum_{k=1}^{K} \frac{h_{ik}}{h_{i1}} z_k \\ &= y \cdot h_{i0} + h_{i1} \cdot s(z). \end{aligned} \tag{122}$$

By Lemma 1, we can express the demand system h as

$$h_0(y,z) = a \cdot y\omega(y, s(z)) \tag{123}$$

and for all $j \neq 0$,

$$h_j(y,z) = \delta_j^A \cdot y\omega(y, s(z)) + \delta_j^B \cdot y(1 - \omega(y, s(z))). \tag{124}$$

By Corollary 2, we can choose a such that $\omega(\overline{y}, s(\overline{z})) \in (0, 1)$. Let

$$G = \left\{(y,z) \in (U \times V)|_{\overline{p}} : \omega(y,z) \in (0,1)\right\}. \tag{125}$$

G is open, because ω is continuous. Let

$$\underline{y}_A = \inf_{(y,z)\in G} y\omega(y,z) \tag{126}$$

$$\underline{y}_B = \inf_{(y,z)\in G} y(1-\omega(y,z)) \tag{127}$$

$$\underline{\omega} = \inf_{(y,z)\in G} \omega(y,z) \tag{128}$$

$$\overline{y}_A = \sup_{(y,z)\in G} y\omega(y,z) \tag{129}$$

$$\overline{y}_B = \sup_{(y,z)\in G} y(1-\omega(y,z)) \tag{130}$$

$$\overline{\omega} = \sup_{(y,z)\in G} \omega(y,z) \tag{131}$$

and define

$$
\begin{aligned}
D_{AB} &= (a\underline{y}_A, a\overline{y}_A) &(132)\\
D_A &= \prod_{j=1}^{m}(\delta_j^A \cdot \underline{y}_A, \delta_j^A \cdot \overline{y}_A) &(133)\\
D_B &= \prod_{j=1}^{m}(\delta_j^B \cdot \underline{y}_B, \delta_j^B \cdot \overline{y}_B). &(134)
\end{aligned}
$$

If any of the δ_j^A or δ_j^B are zero, we may replace $(\delta_j^A \cdot \underline{y}_A, \delta_j^A \cdot \overline{y}_A) = \emptyset$ with $(0, \infty)$, so that D_A and D_B remain open and convex. Finally, let

$$
\begin{aligned}
u_A(q_0, q_A) &= a\log(q_0) + \sum_{j=1}^{m}\delta_j^A \log(q_{Aj}) &(135)\\
u_B(q_0, q_B) &= b\log(q_0) + (1-b)\cdot\sum_{j=1}^{m}\delta_j^B \log(q_{Bj}) &(136)
\end{aligned}
$$

for some $b > 0$ such that

$$b < \frac{a \cdot \underline{\omega}}{(1-\underline{\omega}) + a \cdot \underline{\omega}} \quad (137)$$

so that for all $(y, z) \in G$,

$$
\begin{aligned}
\omega(y,z) &\geq \underline{\omega}\\
&> \frac{b}{a + (1-a)b} &(138)
\end{aligned}
$$

which implies

$$b[(1-\omega(y,z)) + a\omega(y,z)] < a\omega(y,z) \quad (139)$$

so that B will not contribute in equilibrium. Thus, h meets Definition 8, and is consistent with a Cournot model. $\square$

Proof of Proposition 7. As in the proof of Proposition 6, decompose h as

$$h_0(y,z) = a \cdot y\omega(y, s(z)) \quad (140)$$

and for all $j \neq 0$,

$$h_j(y,z) = \delta_j^A \cdot y\omega(y, s(z)) + \delta_j^B \cdot y(1-\omega(y, s(z))). \quad (141)$$

but let

$$
\begin{aligned}
u_A(q_0, q_A) &= a\log(q_0) + \sum_{j=1}^{m}\delta_j^A \log(q_{Aj}) &(142)\\
u_B(q_0, q_B) &= \sum_{j=1}^{m}\delta_j^B \log(q_{Bj}). &(143)
\end{aligned}
$$

Then h meets Definition 7, and is consistent with a collective model with no public goods. $\square$

A.3 Proofs for Section 6

Proof of Proposition 8. Since (p, y) is fixed, I suppress the dependence of all functions on those variables. The demand system is a set of $m+1$ functions $h_i : (0,1) \longrightarrow (0,1)$ such that $\sum_{i=0}^{m} h_i(\omega) = 1$ for all $\omega \in (0,1)$.

If h is generated by a collective model, then $h(\omega) = h^{**}(\mu(\omega))$ for some function $\mu(\omega)$, where

$$\begin{aligned} h^{**}(\mu) &= \arg\max \mu \cdot u_A(q_0, v_A(q_A)) + (1-\mu) \cdot u_B(q_0, v_B(q_B)) \\ &\text{s.t.} \quad q_0 + \sum_{i=0}^{m} (q_{Ai} + q_{Bi}) \leq y. \end{aligned} \tag{144}$$

Suppose $h_0^{**}(\mu)$ were strictly monotone in μ. Note that because $\mu(\cdot)$ is surjective and the support of ω is the entire unit interval $[0,1]$, $\sup_{\omega \in (0,1)} \mu(\omega) = 1$ and $\inf_{\omega \in (0,1)} \mu(\omega) = 0$. Then, letting $\widetilde{\mu} = (h_0^{**})^{-1}(h_0(\widetilde{\omega}))$, we have $\widetilde{\mu} \in \{0,1\}$. That is, the minimum consumption of the public good occurs when one partner has no bargaining power at all. But then that partner must have zero private consumption, too, so

$$\min\{h_1(y, \widetilde{\omega}), h_2(y, \widetilde{\omega})\} = 0. \tag{145}$$

To show that the collective demand for the public good $h_0^{**}(\mu)$ is monotone in μ, let

$$\overline{q}_A = \arg\max v_A(q_A) \text{ s.t. } \sum_{i=1}^{m} q_{Ai} \leq 1 \tag{146}$$

$$\overline{v}_A = v_A(\overline{q}_A) \tag{147}$$

$$\overline{q}_B = \arg\max v_B(q_B) \text{ s.t. } \sum_{i=1}^{m} q_{Bi} \leq 1 \tag{148}$$

$$\overline{v}_B = v_B(\overline{q}_B) \tag{149}$$

The separability assumptions on u_A and u_B and homogeneity assumptions on v_A and v_B mean that the set of efficient private consumption vectors q_A^{**}, q_B^{**} lies in a subspace of $\mathbb{R}^m$ of dimension at most two: that is, any efficient allocation $(q_0^{**}, q_A^{**}, q_B^{**})$ must be of the form $(q_0^{**}, x_A^{**} \cdot \overline{q}_A(1), x_B \cdot \overline{q}_B(1))$, where $q_0^{**} + x_A^{**} + x_B^{**} = y$.

I will show that $\frac{d}{d\mu} h_0^{**}(\mu) \neq 0$ for all μ, which by continuity will imply that h_0^{**} is monotone. Suppose there were some μ such that $\frac{d}{d\mu} h_0^{**}(\mu) = 0$, and let $(q_0^{**}, x_A^{**} \cdot \overline{q}_A, x_B^{**} \cdot \overline{q}_B)$ be the efficient allocation corresponding to μ. Thus,

$$\frac{\partial u_A / \partial q_0}{\partial u_A / \partial v_A} (q_0^{**}, x_A^{**} \overline{v}_A) \cdot \frac{1}{\overline{v}_A} + \frac{\partial u_B / \partial q_0}{\partial u_B / \partial v_B} (q_0^{**}, x_B^{**} \overline{v}_B) \cdot \frac{1}{\overline{v}_B} = 1. \tag{150}$$

Then for a small change dx in the allocation of private expenditures, $(q_0^{**}, (x_A^{**} + dx) \cdot \overline{q}_A, (x_B^{**} - dx) \cdot \overline{q}_B)$ would be efficient, and we would also have

$$\frac{\partial u_A / \partial q_0}{\partial u_A / \partial v_A} (q_0^{**}, (x_A^{**} + dx) \overline{v}_A) \cdot \frac{1}{\overline{v}_A} + \frac{\partial u_B / \partial q_0}{\partial u_B / \partial v_B} (q_0^{**}, (x_B^{**} - dx) \overline{v}_B) \cdot \frac{1}{\overline{v}_B} = 1. \tag{151}$$

Subtracting (150) from (151) and letting $dx \to 0$, we have that $\frac{d}{d\mu} h_0^{**}(\mu) = 0$ implies that

$$\frac{\partial}{\partial v_A}\left(\frac{\partial u_A/\partial q_0}{\partial u_A/\partial v_A}(q_0^{**}, v_A(q_A^{**}))\right) \quad = \quad \frac{\partial}{\partial v_B}\left(\frac{\partial u_B/\partial q_0}{\partial u_B/\partial v_B}(q_0^{**}, v_B(q_B^{**}))\right) \qquad (152)$$

which contradicts (87). □

Proof of Proposition 9. By Proposition 2, the minimum expenditure on the public good occurs at $\widetilde{\omega} \in (\omega_*, \omega^*)$, but $0 < \omega_* < \omega^* < 1$. If the distribution of income is interior, neither A nor B will have zero private consumption, so

$$\min\{h_1(y, \widetilde{\omega}), h_2(y, \widetilde{\omega})\} \quad > \quad 0. \qquad (153)$$

□

B Efficiency Loss in a CES Cournot Model

B.1 Defining the Efficiency Loss

Let (q_0^*, q_A^*, q_B^*) be the equilibrium allocation for the Cournot model. Define the equilibrium utilities

$$u_A^*(\omega|p, y) \quad = \quad u_A(q_0^*(\omega|p, y), q_A^*(\omega|p, y)) \qquad (154)$$
$$u_B^*(\omega|p, y) \quad = \quad u_B(q_0^*(\omega|p, y), q_B^*(\omega|p, y)) \qquad (155)$$

Consider a social planner's cost-minimization problem, given a profile of utilities (u_A, u_B)

$$\begin{aligned} c^{**}(u_A, u_B|p) \quad &= \quad \min_{(q_0, q_A, q_B)} q_0 + p(q_A + q_B) \qquad (156) \\ \text{s.t.} \quad & u_A(q_0, q_A) \geq u_A \\ & u_B(q_0, q_B) \geq u_B \end{aligned}$$

What I will call the "efficiency loss," d, is the relative difference between the social planner's minimized cost of achieving the utilities that agents enjoy in equilibrium, and the total resources in the household:

$$d(\omega|p, y) \quad = \quad 1 - \frac{c^{**}(u_A^*(\omega|p, y), u_B^*(\omega|p, y))|p)}{y} \qquad (157)$$

One way to interpret d is as the compensating variation associated with an exogenous move to the Lindahl prices that give A and B their equilibrium utilities (or, more poetically, a move to Coasian bargaining). Stated differently, it is A and B's aggregate willingness to pay to hire a social planner.[7]

[7]When both partners have homothetic preferences, it can be shown that $c^{**}(u_A^*(\omega|p, y), u_B^*(\omega)|p, y)$ is homogenous of degree one in (u_A, u_B), and thus the efficiency loss d is homogenous of degree zero in $(y_A, y_B) = (y\omega, y(1-\omega))$. In that situation, there is no loss of generality in setting $y_A + y_B = 1$. Moreover, the envelope theorem and Roy's identity imply that $c^{**}(u_A^*(\omega|p, y), u_B^*(\omega|p, y))$ is locally constant in p (except when variations in p, for constant y, changes the set of equilibrium contributors to the public good), so that d depends only on the relative endowment ω, but not on prices p.

In the following section, I provide explicit formulae for a quadratic approximation to $d(\omega|p, y)$ when both agents have constant-elasticity-of-substitution (CES) preferences.

B.2 A "Harberger Triangle" for the Cournot Model

Let us rewrite the problem (156) in one dimension: define $\widehat{q}_A(q_0, u_A^*(\omega|p, y))$ and $\widehat{q}_B(q_0, u_B^*(\omega|p, y))$ implicitly by

$$u_A^*(\omega|p, y) = u_A(q_0, \widehat{q}_A) \tag{158}$$
$$u_B^*(\omega|p, y) = u_B(q_0, \widehat{q}_B) \tag{159}$$

and let

$$c(q_0) = q_0 + p(\widehat{q}_A(q_0, u_A^*(\omega|p, y)) + \widehat{q}_B(q_0, u_B^*(\omega|p, y))) \tag{160}$$

Let q_0^{**} be the socially optimal quantity of the public good, i.e. the value of q_0 that minimizes (160). Obviously, $c(q_0^*(\omega|p, y)) = y$, and

$$\begin{aligned} c(q_0^{**}) &= c^{**}(u_A^*(\omega|p, y), u_B^*(\omega|p, y)|p)) \\ &= y(1 - d(\omega|p, y)) \end{aligned} \tag{161}$$

In general, $d(\omega|p, y)$ must be calculated numerically. However, a useful shortcut is to take a quadratic approximation to the social planner's cost function (160) about q_0^{**}. Doing so leads to the following version of a Harberger triangle:

Proposition 10. *The equilibrium efficiency loss is*

$$d(\omega|p, y) - \frac{1}{2}\frac{[c'(q_0^*)]^2}{(y_A + y_B)c''(q_0^*)} = O\left((q_0^* - q_0^{**})^2\right). \tag{162}$$

It remains to calculate the first and second derivatives $c'(q_0^*)$ and $c''(q_0^*)$. As a shorthand, let

$$MRS(q_0^*, u_A^*) = \frac{\partial u_A/\partial q_0}{\partial u_A/\partial q_{A1}}(q_0^*, q_{A1}^*) \tag{163}$$

be A's marginal willingness to pay for the public good at the Cournot equilibrium. Then the the numerator in (162) is

$$\begin{aligned} c'(q_0^*) &= 1 + p\left(\frac{\partial \widehat{q}_A}{\partial q_0} + \frac{\partial \widehat{q}_B}{\partial q_0}\right) \\ &= 1 - p\left(MRS(q_0^*, u_A^*) + MRS(q_0^*, u_B^*)\right) \end{aligned} \tag{164}$$

By construction, the first derivative c' is the difference between the relative price of the public good and the sum of A and B's marginal rates of substitution, i.e. the marginal social willingness to pay for the public good. The efficiency loss is therefore approximately quadratic in the marginal distortion in equilibrium, echoing the familiar analysis of the deadweight loss of taxation.

When both partners have constant-elasticity of substitution (CES) preferences, i.e.

$$u_A(q_0, q_{A1}) = (aq_0^{\rho_A} + (1-a)q_{A1}^{\rho_A})^{1/\rho_A} \tag{165}$$

$$u_B(q_0, q_{B1}) = (bq_0^{\rho_B} + (1-b)q_{B1}^{\rho_B})^{1/\rho_B} \tag{166}$$

the first derivative $c'(q_0^*)$ in (164) becomes

$$c'(q_0^*) = 1 - p\left[\left(\frac{a}{1-a}\right)\left(\frac{q_0^*}{q_A^*}\right)^{\rho_A - 1} + \left(\frac{b}{1-b}\right)\left(\frac{q_0^*}{q_B^*}\right)^{\rho_B - 1}\right] \tag{167}$$

Obtaining the denominator $c''(q_0^*)$ takes more work, but is not complicated. Note that A's marginal willingness to pay for q_0 is exactly

$$MRS(q_0^*, u_A^*) = \frac{1}{\widehat{p}(q_0^*, u_A^*)} \tag{168}$$

where $\widehat{p}(q_0, u)$ is A's inverse Hicksian demand for q_0. Then

$$c''(q_0^*) = -p\left[\frac{1}{[\widehat{p}(q_0^*, u_A^*)]^2}\frac{\partial\widehat{p}(q_0^*, u_A^*)}{\partial q_0} + \frac{1}{[\widehat{p}(q_0^*, u_B^*)]^2}\frac{\partial\widehat{p}(q_0^*, u_B^*)}{\partial q_0}\right] \tag{169}$$

Assembling the components of (169), we have A's Hicksian demand given by

$$q_0^H(p, u) = u \cdot a^{\sigma_A} \cdot \left[a^{\sigma_A} + (1-a)^{\sigma_A} p^{1-\sigma_A}\right]^{\frac{\sigma_A}{1-\sigma_A}} \tag{170}$$

where, as usual, we let $\sigma_A = (1-\rho_A)^{-1}$ and $\sigma_B = (1-\rho_B)^{-1}$ denote A and B's elasticities of substitution between the private and public goods.

Then,

$$\widehat{p}(q_0, u_A^*) = \left(\frac{a}{1-a}\right)^{\frac{\sigma_A}{1-\sigma_A}}\left[\frac{1}{a}(u_A^*)^{\frac{\sigma_A - 1}{\sigma_A}} q_0^{\frac{1-\sigma_A}{\sigma_A}} - 1\right]^{\frac{1}{1-\sigma_A}} \tag{171}$$

Differentiating, we obtain (after some algebra)

$$\frac{\partial\widehat{p}(q_0, u_A^*)}{\partial q_0} = \frac{1}{\sigma_A q_0}\left[\widehat{p}(q_0, u_A^*) + \left(\frac{a}{1-a}\right)^{\sigma_A}\widehat{p}(q_0, u_A^*)^{\sigma_A}\right] \tag{172}$$

Repeating the analogous calculations for B and substituting in (169), we finally arrive at

$$\begin{aligned} c''(q_0^*) &= \frac{p}{q_0^*}\left\{\frac{1}{\sigma_A}\left(MRS_A(q_0^*, u_A^*) + \left[\frac{a}{1-a}\right]^{\sigma_A} MRS_A(q_0^*, u_A^*)^{2-\sigma_A}\right)\right. \\ &+ \left.\frac{1}{\sigma_B}\left(MRS_B(q_0^*, u_B^*) + \left[\frac{b}{1-b}\right]^{\sigma_B} MRS_B(q_0^*, u_B^*)^{2-\sigma_B}\right)\right\} \end{aligned} \tag{173}$$

Example 5. Suppose both partners have Cobb-Douglas preferences as in Examples 3 and 1 above, and also as before let the aggregate endowment y be unity. In Figure 3, I illustrate the approximate efficiency loss as a function of the distribution of endowments, ω.

The efficiency loss tends to be highest when the distribution of wealth, ω, is fairly equal. This is because when one partner free-rides and does not contribute to the public good, their willingness to pay for q_0 must be strictly lower than its relative price. However, when $\omega \in (\omega_*, \omega^*)$, both A and B contribute to the public good. Thus, both A and B's marginal willingness to pay for q_0 is equal to its relative price, and the gap between the social marginal cost of q_0 and the social willingness to pay - the right-hand side of (49) - is maximal. $\triangle$

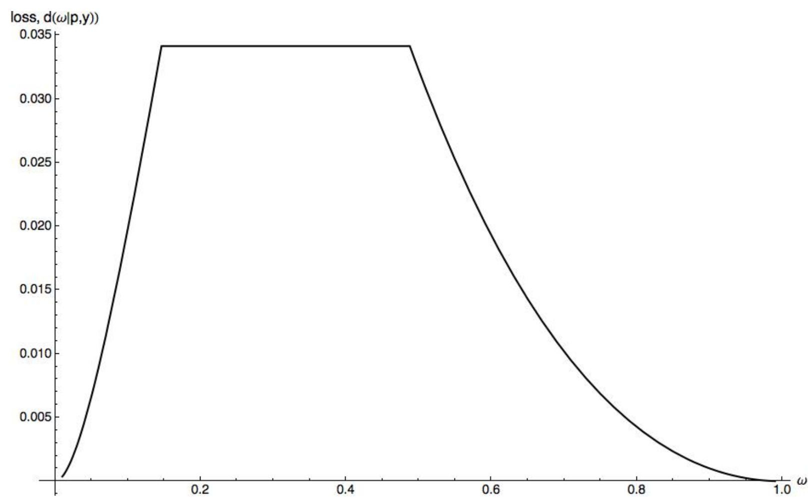

Figure 3: Efficiency losses, $d(\omega|p, y)$, under Cobb-Douglas preferences. Share parameters are $a = 0.7$ and $b = 0.4$. The relative prices of the private goods are $p_1 = p_2 = 1$, and total household wealth is $y = 1$.

B.3 Dependence of Efficiency Loss on Preferences

Of course, the efficiency loss $d(\omega|p, y)$ depends on the preferences of the two agents as well. In Figures 4 and 5 I illustrate this dependence for the share parameters (a, b) and the substitution elasticities (σ_A, σ_B) separately.

With the numerical values I have chosen, efficiency losses tend to be somewhat small (peaking at about 4% of household wealth), but it is possible to drive the maximum loss up to over 10% of household wealth by making the public and private good very substitutable - with σ_A and σ_B equal to 2.5 or higher - and making the share parameters a and b close to 50%.

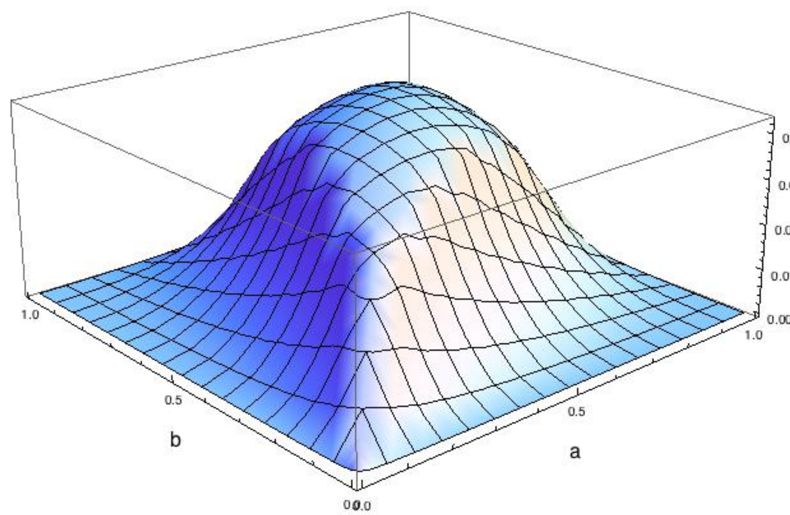

Figure 4: Efficiency losses, $d(\omega|p, y)$, as a function of agents' preferences, a and b. The agents' substitution elasticities σ_A and σ_B are fixed at 1.0. Relative prices and the aggregate endowment are fixed at $p = 1$ and $y = 1$. A's relative endowment, ω, is fixed at 0.5.

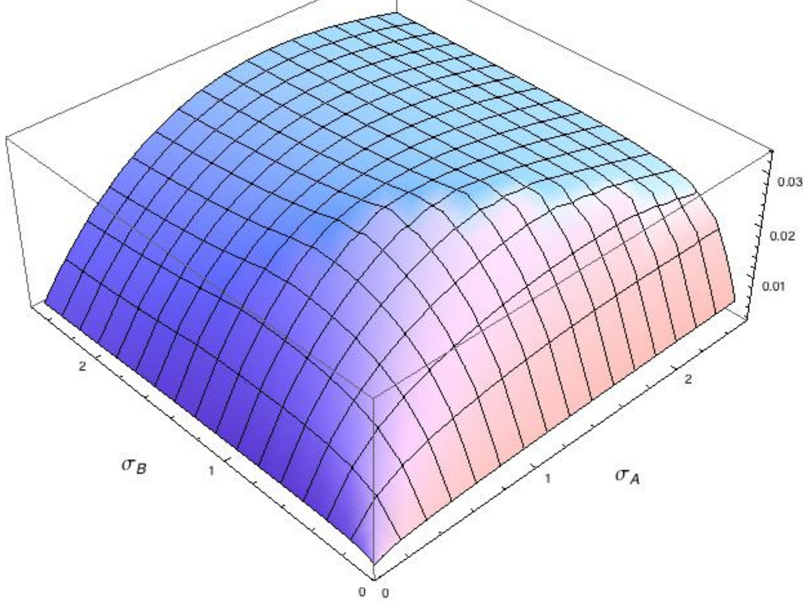

Figure 5: Efficiency losses, $d(\omega|p, y)$, as a function of agents' preferences, σ_A and σ_B. The agents' share parameters are fixed at $a = 0.7$ and $b = 0.4$. Relative prices and the aggregate endowment are fixed at $p = 1$ and $y = 1$. A's relative endowment, ω, is fixed at 0.5.

C Implications of Efficiency With Price Variation

None of the following is original; it is simply a repetition of material from [Browning & Chiappori, 1998], included for completeness only.

DEFINITION 9. Let $g \in \mathcal{G}(U, V)$ be an extended aggregate demand system, and let $(p, y, z) \in U \times V$. If the "pseudo-Slutsky" matrix $S(p, y, z)$, with (i, j)-th entry

$$s_{ij}(p, y, z) = \frac{\partial g_i}{\partial p_j} + g_j \cdot \frac{\partial g_i}{\partial y} \tag{174}$$

can be written as

$$S(p, y, z) = \Sigma(p, y, z) + R(p, y, z) \tag{175}$$

for some negative semidefinite matrix Σ and some matrix R with $\text{rank}(R) \leq 1$, then we say that g is *negative semidefinite plus rank one at* (p, y, z), or, as a shorthand, g *is SR1 at* (p, y, z).

PROPOSITION 11. *If the extended aggregate demand system g is induced by a collective model $\theta \in \Theta(U, V)$, then g is SR1 at all $(p, y, z) \in U \times V$.*

Proof. Since g is induced by a collective model $\theta = (\mathcal{I}, u_A, u_B, \mu)$, we can write

$$g(p, y, z) = g^{**}(p, y, \mu(p, y, z)|\theta) \tag{176}$$

Differentiating (176) for any good i, we have

$$\frac{\partial g_i}{\partial p_j} = \frac{\partial g_i^{**}}{\partial p_j} + \frac{\partial g_i^{**}}{\partial \mu} \cdot \frac{\partial \mu}{\partial p_j} \tag{177}$$

$$\frac{\partial g_i}{\partial y} = \frac{\partial g_i^{**}}{\partial y} + \frac{\partial g_i^{**}}{\partial \mu} \cdot \frac{\partial \mu}{\partial y} \tag{178}$$

$$\tag{179}$$

so that

$$\begin{aligned} s_{ij}(p, y, z) &= \frac{\partial g_i}{\partial p_j} + g_j \cdot \frac{\partial g_i}{\partial y} \\ &= \frac{\partial g_i^{**}}{\partial p_j} + \frac{\partial g_i^{**}}{\partial \mu} \cdot \frac{\partial \mu}{\partial p_j} + g_j^{**} \left[\frac{\partial g_i^{**}}{\partial y} + \frac{\partial g_i^{**}}{\partial \mu} \cdot \frac{\partial \mu}{\partial y} \right] \\ &= \left\{ \frac{\partial g_i^{**}}{\partial p_j} + g_j^{**} \cdot \frac{\partial g_i^{**}}{\partial y} \right\} + \frac{\partial g_i^{**}}{\partial \mu} \cdot \left\{ \frac{\partial \mu}{\partial p_j} + g_j^{**} \cdot \frac{\partial \mu}{\partial y} \right\} \end{aligned} \tag{180}$$

The first term in the final line of (180) is the (i, j)-th entry in the Slutsky matrix of the demand function $g^{**}(p, y, \mu(p, y, z))$, *holding μ fixed.* If it were possible to observe demands by varying (p, y) without also varying the Pareto weight μ, the result would be the constrained maximizer of the concave function $\mu \cdot u_A + [1 - \mu]u_B$. Thus, the matrix Σ with such entries is negative semidefinite.

The second term in (180) is the the (i, j)-th entry in the matrix

$$R \quad = \quad \begin{bmatrix} \frac{\partial g_0^{**}}{\partial \mu} \\ \frac{\partial g_1^{**}}{\partial \mu} \\ \vdots \\ \frac{\partial g_m^{**}}{\partial \mu} \end{bmatrix} \times \left[\frac{\partial \mu}{\partial p_0} + g_0^{**} \frac{\partial \mu}{\partial y}, \quad \frac{\partial \mu}{\partial p_1} + g_1^{**} \frac{\partial \mu}{\partial y}, \quad \ldots \quad \frac{\partial \mu}{\partial p_m} + g_m^{**} \frac{\partial \mu}{\partial y} \right] \tag{181}$$

Since R is the outer product of two vectors, $\mathrm{rank}(R) \leq 1$. $\square$

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